FLRW models: Kinematics, dynamics, geometry

Cosmology Block Course 2014

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Contents

Cosmic time and FLRW metric

Friedmann-Robertson-Walker-Metric: Substrate of “galaxy dust” moving with the cosmic flow (scale factor expansion). Distances between substrate galaxies change proportionally to the cosmic scale factor $a(t)$. Time coordinate t (cosmic time) is proper time of substrate particles. Different kinds of possible geometry parametrized by $K = -1, 0, +1$ (hyperbolic, flat, spherical):

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega\right] = -c^2\mathrm{d}\tau^2.$$

with $\mathrm{d}\Omega \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

Taylor expansion of the scale factor

Generic Taylor expansion:

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a} \cdot (t - t_0)^2 + \ldots$$

Re-define the expansion parameters by introducing two functions

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \quad \text{and} \quad q(t) \equiv -\frac{\ddot{a}(t)a(t)}{\dot{a}(t)^2}$$

and corresponding constants

$$H_0 \equiv H(t_0) \quad \text{and} \quad q_0 \equiv q(t_0)$$

$$a(t) = a_0 \left[1 + (t - t_0)H_0 - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \ldots \right]$$

Some nomenclature and values 1/2

t_0 is the standard symbol for the **present time**. If coordinates are chosen so cosmic time $t = 0$ denotes the time of the big bang (phase), then t_0 is the **age of the universe**. Sometimes, the age of the universe is denoted by τ.

$H(t)$ is the **Hubble parameter** (sometimes misleadingly *Hubble constant*)

$H_0 \equiv H(t_0)$ is the **Hubble constant**. Current values are around

$$H_0 = 70\,\frac{\mathrm{km/s}}{\mathrm{Mpc}} \approx 10^{-10}/a$$

Some nomenclature and values 2/2

Sometimes, the Hubble constant is written as

$$H_0 = h \cdot 100\ \frac{\mathrm{km/s}}{\mathrm{Mpc}}$$

to keep one's options open with h the **dimensionless Hubble constant**.

The inverse of the Hubble constant is the **Hubble time** (cf. the linear case and the models later on).

$$\frac{1}{h \cdot 100 \frac{\mathrm{km/s}}{\mathrm{Mpc}}} \approx h^{-1} \cdot 10^{10}\ \mathrm{a}.$$

Light in an FRW universe

For light, often easier to use $\mathrm{d}s^2 = 0$ instead of the geodetic equation.

Also: use symmetries! Move origin of your coordinate system wherever convenient. Look only at radial movement.

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega\right]$$

becomes

$$\mathrm{d}t = \pm\frac{a(t)\,\mathrm{d}r/c}{\sqrt{1-Kr^2}}.$$

Light in an FRW universe

Integrate to obtain

$$c\int_{t_2}^{t_1}\frac{\mathrm{d}t}{a(t)} = \pm\int_{r_2}^{r_1}\frac{\mathrm{d}r}{\sqrt{1-Kr^2}}.$$

Plus/minus: light moving towards us or away from us.

The key to astronomical observations in an FRW universe:

$$c\int_{t_1}^{t_0}\frac{\mathrm{d}t}{a(t)} = \int_{0}^{r_1}\frac{\mathrm{d}r}{\sqrt{1-Kr^2}}.$$

where, by convention, t_0 is present time, $t_1 < t_0$ emission time of particle, r_1 (constant) coordinate value for distant source.

Light signals chasing each other 1/2

Imagine two signals leaving a distant galaxy at $r = r_1$ at consecutive times t_1 and $t_1 + \delta t_1$, arriving at t_0 and δt_0. Then

$$\int_{t_1}^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{1-Kr^2}}.$$

and

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{c\,\mathrm{d}t}{a(t)} = \int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{1-Kr^2}}$$

$$\int_{t_0}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)} - \int_{t_1}^{t_1+\delta t_1} \frac{\mathrm{d}t}{a(t)} = 0.$$

Light signals chasing each other 2/2

For small δt,

$$\int_{\bar{t}}^{\bar{t}+\delta t} f(t)\,\mathrm{d}t \approx f(\bar{t}) \cdot \delta t,$$

so in our case

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

Signals could be anything — in particular: consecutive crests (or troughs) of elementary light waves of frequency $f \propto 1/\delta t$:

$$\frac{f_0}{f_1} = \frac{a(t_1)}{a(t_0)}, \text{wavelengths change as } \frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)}.$$

Frequency shift by expansion

Frequency shift z (commonly *redshift*) defined as

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(t_0)}{a(t_1)} - 1$$

or

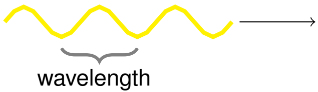
$$1 + z = \frac{a(t_0)}{a(t_1)}$$

For co-moving galaxies: z is directly related to r_1. For monotonous $a(t)$: distance measure.

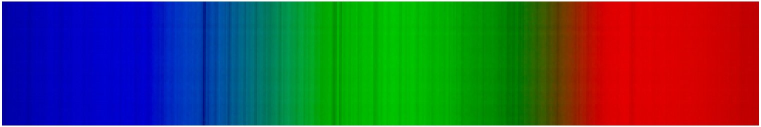
Relation depends on dynamics ⇒ later!

Cosmological redshift

Wavelength scaling with scale factor:



Redshift for $a(t_0) > a(t_1)$; blueshift for $a(t_0) < a(t_1)$



For “nearby” galaxies. . .

. . . use the Taylor expansion $a(t) = a(t_0)[1 + H_0(t - t_0) + O((t - t_0)^2)]$:

$$1 - z \approx \frac{1}{1+z} = \frac{a(t_1)}{a(t_0)} \approx 1 + H_0(t_1 - t_0)$$

or

$$z \approx H_0(t_0 - t_1) \approx H_0 d/c$$

for small z, small $t_0 - t_1$, d the distance of the galaxy from us

This is Hubble’s law.
Originally discovered by Alexander Friedmann (cf. Stigler’s law).

Pedestrian derivation of Hubble's law and redshift

For scale factor expansion, $d(t) = a(t)/a(t_0) \cdot d(t_0)$:
"Instantaneous speed" of a galaxy

$$v(t) = \frac{\dot{a}(t)}{a(t)} d(t) = H(t)\, d(t) \approx H_0\, d(t).$$

Classical (moving-source) Doppler effect:

$$cz = v$$

in other words:

$$cz = H_0\, d.$$

"Speed of recession" – we'll come back later to the question whether or not that is a real speed.

Solving Einstein's equations for FRW

00 component of Einstein's eq.:

$$3\,\frac{\dot{a}^2 + Kc^2}{a^2} = 8\pi G\,\rho$$

$i0$ components vanish. ij components give

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + Kc^2}{a^2} = -\frac{8\pi G}{c^2}p$$

These are the **Friedmann equations**. Their solutions are the **Friedmann-Lemaître-Robertson-Walker** (FLRW) universes.

Re-casting the Friedmann equations

Take what we will call the *first-order Friedmann equation*

$$\frac{\dot{a}^2 + Kc^2}{a^2} = \frac{8\pi G\rho}{3} \tag{1}$$

and for $\dot{a} \neq 0$, differentiating the above and inserting the ij-equation, derive

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p/c^2) = -3H(t)(\rho + p/c^2).$$

— this amounts to energy conservation (as you've seen in Monday's exercise).

The physics behind the Friedmann equations

Multiply

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p/c^2)$$

by a^3c^2 and integrate:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho c^2\, a^3) + p\frac{\mathrm{d}a^3}{\mathrm{d}t} = 0.$$

The volume of a small ball $0 \leq r \leq r_1$ is

$$V = \iiint \sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = a^3\; v(r_1).$$

Using this, rewrite

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho c^2 V) + p\frac{\mathrm{d}V}{\mathrm{d}t} = 0.$$

The physics behind the Friedmann equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho c^2 V) + p\frac{\mathrm{d}V}{\mathrm{d}t} = 0.$$

but ρc^2 is energy density — $\rho V = U$ is the system's energy!

$$\Rightarrow \quad \mathrm{d}E = -p\,\mathrm{d}V$$

— change in energy is the "expansion work".

If $p = 0$ (dust universe), $\mathrm{d}E = 0$, so energy/mass is conserved!

Physics behind the Friedmann: deceleration

Recombine Friedmann equations to give equation for $\ddot{a}$ (which we'll call the second-order Friedmann equation):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p/c^2) \tag{2}$$

Almost Newtonian — but in general relativity, pressure is a source of gravity, as well! (E.g. stellar collapse.)

This leads to an expression for the deceleration parameter:

$$q_0 = \frac{4\pi G}{3}(\rho_0 + 3p_0/c^2)$$

(with ρ_0 and p_0 the present density/pressure).

Newtonian analogy

Using purely Newtonian reasoning, one can derive the Friedmann equations for dust for $K = 0$ or, with a slight modification for the source terms of Newtonian gravity, also for matter with pressure.

Details $\Rightarrow$ **You've done this in Monday's exercise**

Different equations of state

Now, assume equation of state $p = w\rho c^2$. Then

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p/c^2)$$

becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

which is readily integrated to

$$\rho \sim a^{-3(1+w)}.$$

This describes how the cosmic content is *diluted* by expansion.

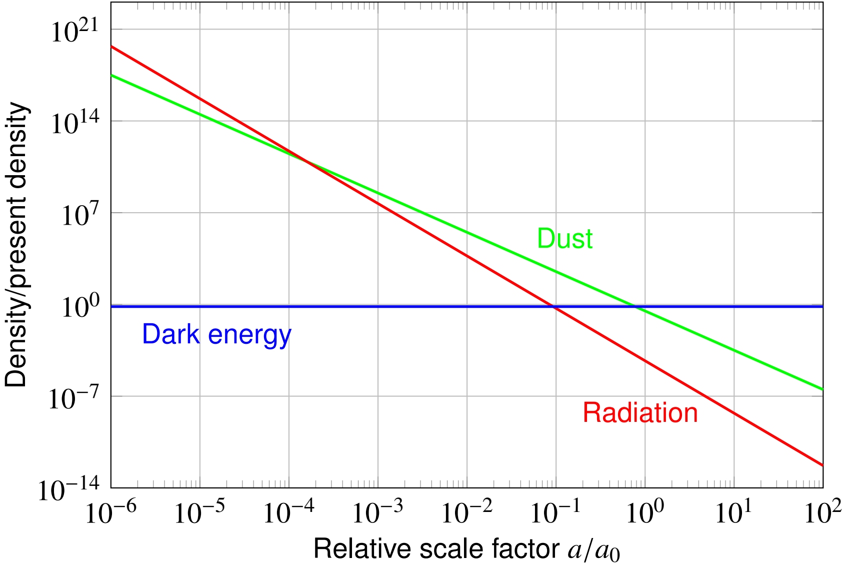
How does density change with the scale factor?

Earlier on, we had looked a three different equations of state $p = w\rho c^2$:

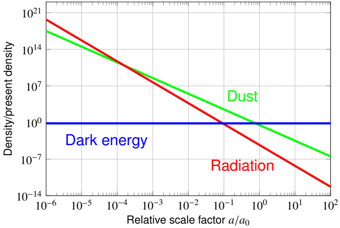
1. **Dust**: $w = 0 \quad \Rightarrow \quad \rho \sim 1/a^3$
2. **Radiation**: $w = 1/3 \quad \Rightarrow \quad \rho \sim 1/a^4$
3. **Scalar field/dark energy**: $w = -1 \;\; \rho =$const.

Whenever these are the only important components, a universe can have different *phases* — depending on size, different components will dominate.

Different eras depending on the scale factor



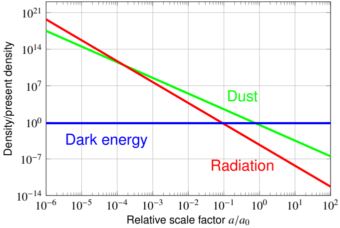
Different eras depending on the scale factor



Two caveats:

- This says little about evolution — some values of a might not even be reached
- In reality, matter will change — particles might start as dust (non-relativistic) and, at smaller a, end up at high energies and thus as radiation (relativistic particles)

For small a: Radiation dominates!



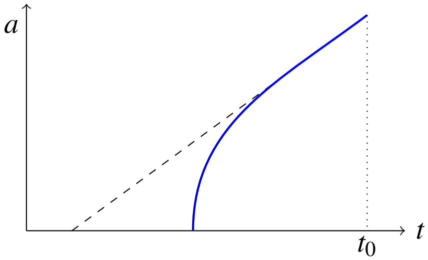
- If sufficiently small values of a are reached, radiation contribution (including relativistic particles) will dominate everything else
- This will be the basis of our models for the early universe (convenient — no need to worry about dust and dark energy at early times!)

The initial singularity

We derived the second-order Friedmann equation as

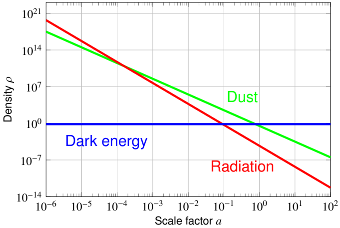
$$3\frac{\ddot{a}}{a} = -\frac{4\pi G}{c^4}(\rho + 3p/c^2) = -\frac{4\pi G}{c^4}(\rho_{\text{dust}} + 2\,\rho_{\text{rad}} - 2\,\rho_\Lambda)$$

For universes where Λ does not dominate completely, $\ddot{a}/a \leq 0$:



Initial singularity — special case of Hawking-Penrose theorems

If a $\Lambda > 0$ universe becomes large, Λ dominates



Remember the deceleration parameter:

$$q_0 = \frac{4\pi G}{3}(\rho_0 + 3p_0/c^2).$$

Occasional misunderstanding: “Dark energy is negative, and acts like negative mass” – no: what accelerates the expansion is the negative pressure, $p_\Lambda = -\rho_\Lambda c^2$, and the factor 3!

Introducing the critical density

Evaluate the Friedmann equation

$$3\,\frac{\dot{a}^2 + Kc^2}{a^2} = 8\pi G\rho$$

at the present time t_0 to obtain

$$1 = \frac{8\pi G}{3H_0^2}\rho_0 - \frac{Kc^2}{a_0^2 H_0^2}.$$

Where $\rho_0 \equiv \rho(t_0)$. The expression

$$\rho_{c0} \equiv \frac{3H_0^2}{8\pi G}$$

is called the *critical density* (at the present time).

Critical density and geometry

The present-time Friedmann equation then becomes

$$\rho_0/\rho_{c0} = 1 + \frac{Kc^2}{a_0^2 H_0^2}.$$

This equation links the present energy (mass) density ρ_0 of the universe with the Hubble constant H_0 (partly disguised as ρ_{c0}) and the geometry K:

$\rho_0 > \rho_{c0}$	$\Leftrightarrow$	$K = +1$	spherical space
$\rho_0 = \rho_{c0}$	$\Leftrightarrow$	$K = 0$	Euclidean space
$\rho_0 < \rho_{c0}$	$\Leftrightarrow$	$K = -1$	hyperbolical space

Misconception about critical density and geometry

$\rho_0 > \rho_{c0}$	$\Leftrightarrow$	spherical,	finite,	cosmos will collapse
$\rho_0 = \rho_{c0}$	$\Leftrightarrow$	Euclidean,	infinite,	cosmos will keep expanding
$\rho_0 < \rho_{c0}$	$\Leftrightarrow$	hyperbolical,	infinite,	cosmos will keep expanding

Synonyms: finite = “closed universe”, infinite = “open universe”.

- Local geometry does not control topology!
- Direct correspondence with collapse or not only for $\Lambda = 0$!

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Parametrizing simple FLRW models

Assume that there is no interaction between dust (which we will now call simply “matter”, index M), radiation (index R) and dark energy (index Λ): Densities and pressures just add up.

Rescale all present densities in terms of the present critical density, and re-scale K accordingly:

$$\Omega_{\Lambda 0} = \rho_\lambda(t_0)/\rho_{c0}, \quad \Omega_{m0} = \rho_M(t_0)/\rho_{c0},$$
$$\Omega_{r0} = \rho_R(t_0)/\rho_{c0}, \quad \Omega_{K0} = -Kc^2/(a_0 H_0)^2.$$

Present-day Friedmann equation becomes

$$\Omega_{\Lambda 0} + \Omega_{m0} + \Omega_{r0} + \Omega_{K0} = 1.$$

This is how the densities are linked with spatial geometry.

Re-writing the Friedmann equations in terms of the Ωs

Scaling behaviour of the different densities means that

$$\rho(t) = \frac{3H_0^2}{8\pi G}\left[\Omega_{m0}\left(\frac{a_0}{a(t)}\right)^3 + \Omega_{r0}\left(\frac{a_0}{a(t)}\right)^4 + \Omega_{\Lambda 0}\right].$$

Define $x(t) \equiv a(t)/a_0 = 1/(1+z)$ to rewrite the first-order Friedmann equation (1) as

$$H(t)^2 = H_0^2\left[\Omega_{\Lambda 0} + \Omega_{K0}\, x^{-2} + \Omega_{m0}\, x^{-3} + \Omega_{r0}\, x^{-4}\right].$$

General considerations for FLRW models

Re-cast the previous equation as

$$\mathrm{d}t = \frac{\mathrm{d}x}{H_0\, x\, \sqrt{\Omega_{\Lambda 0} + \Omega_{K0}\, x^{-2} + \Omega_{m0}\, x^{-3} + \Omega_{r0}\, x^{-4}}}$$

which governs the time dependence of the cosmic scale factor!

The age of the universe in FLRW models

Simple application: Choose $t = 0$ by $a(0) = 0$ [cosmic time starts at initial singularity]. Integrate to the present time (which has $x = 1$ and $t = t_0$). This gives t_0, the age of the universe:

$$t_0 = \frac{1}{H_0} \int\limits_0^1 \frac{\mathrm{d}x}{x\,\sqrt{\Omega_{\Lambda 0} + \Omega_{K0}\, x^{-2} + \Omega_{m0}\, x^{-3} + \Omega_{r0}\, x^{-4}}}.$$

The acceleration (ex deceleration) parameter q_0

Present pressure:

$$p_0 = \frac{3H_0^2}{8\pi G}(-\Omega_{\Lambda 0} + \frac{1}{3}\Omega_{r0}).$$

inserting in

$$q_0 = \frac{4\pi G}{3}(\rho_0 + 3p_0),$$

we find that

$$q_0 = \frac{1}{2}(\Omega_{m0} - 2\Omega_{\Lambda 0} + 2\Omega_{r0}).$$

The fate of FLRW universes

Rewrite

$$\frac{\dot{a}^2 + Kc^2}{a^2} = \frac{8\pi G\rho}{3c^4}$$

as

$$\dot{a}^2 = (H_0 a_0)^2 \left[\Omega_{\Lambda 0} x^2 + \Omega_{m0} x^{-1} + \Omega_{r0} x^{-2} + \Omega_{K0}\right]$$

where $x = a/a_0$.

If we want a re-collapse, we must have $\dot{a} = 0$ at some time, in other words:

$$\Omega_{\Lambda 0} x^2 + \Omega_{m0} x^{-1} + \Omega_{K0} + \Omega_{r0} x^{-2} = 0.$$

The fate of FLRW universes

Consider universes with negligible Ω_{r0} and with turning point at x that is not too small — realistic for universes that become large and matter-dominated.

Then the "collapse condition" becomes

$$\Omega_{\Lambda 0}\, x^3 + \Omega_{m0} + \Omega_{K0}\, x = 0$$

From present-day Friedmann equation: for $x = 1$, the LHS is $+1$

The fate of FLRW universes

Conclusions from

$$\Omega_{\Lambda 0}\,x^3 + \Omega_{m0} + \Omega_{K0}\,x = 0 \quad \text{where} \quad \Omega_{K0} = -K/(a_0 H_0)^2 :$$

- For $\Omega_{\Lambda 0} < 0$, for sufficiently large x, the expression will become negative $\Rightarrow$ must have a zero
- For $\Omega_{\Lambda 0} = 0$, and since $\Omega_{m0} > 1$, recollapse requires $K = +1$
- For $\Omega_{\Lambda 0} > 0$, recollapse if Ω_{K0} sufficiently negative (again, $K = +1$).

Examples for FLRW universes

Fairly complete classifications exist.

We will study special cases only.

Overview of FLRW solutions for $\Omega_{r0} = 0$

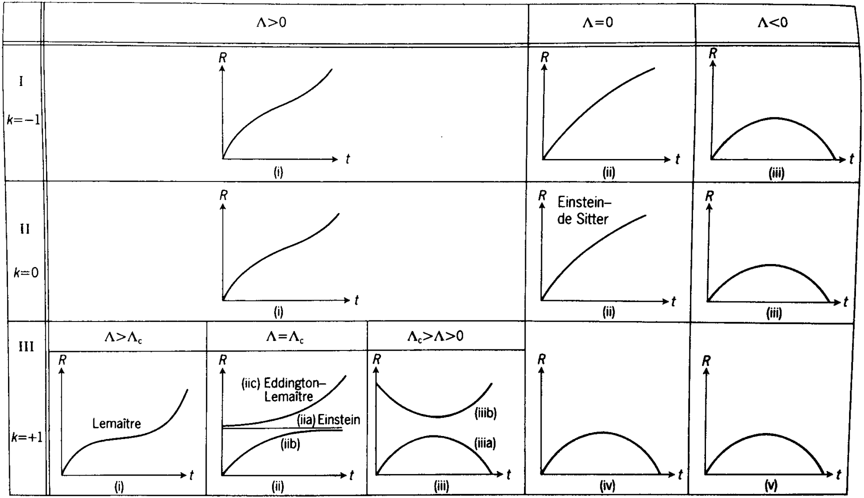


Image from: d'Inverno, *Introducing Einstein's Relativity*, ch. 22.3

Flat radiation universe

Only non-zero contribution: $\Omega_{r0} = 1$ (density must be critical because of flatness)

$$H_0 \int_0^t \mathrm{d}t = \int_0^x x\,\mathrm{d}x$$

so that

$$a = a_0\sqrt{2H_0\,t} \propto \sqrt{t}.$$

Age of the universe:

$$t_0 = \frac{1}{2H_0}$$

Flat matter universe

$$H_0 \int_0^t \mathrm{d}t = \int_0^x \sqrt{x}\,\mathrm{d}x$$

so that

$$a = a_0\,(3/2 \cdot H_0\ t)^{2/3} \propto t^{2/3}$$

Age of the universe:

$$t_0 = \frac{2}{3H_0}$$

de Sitter universe

In this case, the only contribution is the cosmological constant $\Omega_{\Lambda 0}$. We assume space to be flat, so $\Omega_{\Lambda 0} = 1$.

The integrand diverges for $x = 0$, so we choose as an integration boundary the present time t_0, for which $x = 1$:

$$H_0 \int_{t_0}^{t} \mathrm{d}t = \int_{1}^{x} \frac{\mathrm{d}x}{x\sqrt{\Omega_{\Lambda 0}}} = \ln(x)$$

so that

$$a = a_0 \exp(H_0[t - t_0])$$

For this model, de Sitter space-time, the universe is infinitely old — there is no time at which the scale factor is exactly zero.

Light propagation / distances in FLRW spacetimes

Key to astrophysics: Distance determinations (distance ladder)!

- Deduce distance from known length scale (e.g. parallax)
- Deduce distance from known luminosity (standard candle methods)

Both involve the geometry of space. Are they influenced by universal expansion, as well?

Comoving and proper distance

In a FLRW universe:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega\right] = -c^2\mathrm{d}t^2 + a(t)^2\tilde{g}(\vec{x})_{ij}\mathrm{d}x^i\,\mathrm{d}x^j$$

In this coordinate system, galaxy locations *up to scale* can be described by radial coordinate values: **comoving distance**. Good to keep track of where galaxies go! (But: usually dimensionless; order, not length.)

“Instantaneous distances”: stop the universe and measure with a ruler. These are the distances at a fixed time as described by the spatial part of the metric: **proper (spatial) distance**. But: Depends a lot on coordinate choice (cf. Milne universe, later)!

Also, neither are directly measurable.

Co-moving distances related to redshift

From FRLW metric and $\mathrm{d}s^2 = 0$, for light propagation

$$\int \frac{c\,\mathrm{d}t}{a(t)} = \pm \int \frac{\mathrm{d}r}{\sqrt{1-Kr^2}} = \pm \begin{cases} \arcsin(r) & K = +1 \\ r & K = 0 \\ \operatorname{arsinh}(r) & K = -1 \end{cases}.$$

Consider a source at radial coordinate $r(z)$ whose light reaches us with redshift z (using the Friedmann equation with $x = a(t)/a_0$):

$$\int\limits_{t(z)}^{t_0} \frac{c\,\mathrm{d}t}{a(t)} = \frac{c}{a_0 H_0} \int\limits_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\,\sqrt{\Omega_{\Lambda 0} + \Omega_{K0}\,x^{-2} + \Omega_{m0}\,x^{-3} + \Omega_{r0}\,x^{-4}}}.$$

Co-moving distances related to redshift

$$
\begin{aligned}
r(z) &= S\left[\int_{t(z)}^{t_0} \frac{d\,\mathrm{d}t}{a(t)}\right] \\
&= S\left[\frac{c}{a_0 H_0}\int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\,\sqrt{\Omega_{\Lambda 0} + \Omega_{K0}\,x^{-2} + \Omega_{m0}\,x^{-3} + \Omega_{r0}\,x^{-4}}}\right]
\end{aligned}
$$

where

$$
S[y] \equiv \begin{cases} \sin y & K = +1 \\ y & K = 0 \\ \sinh y & K = -1 \end{cases}
$$

Proper distance related to redshift

Use

$$\Omega_{K0} = -\frac{Kc^2}{a_0^2 H_0^2}$$

and $\sinh ix = i \sin x$ to re-write as

$$d_{\mathrm{now}}(z) = a_0 r(z)$$

$$= \frac{c}{H_0 \sqrt{\Omega_{K0}}} \cdot \sinh\left[\sqrt{\Omega_{K0}} \int\limits_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2 \sqrt{\Omega_{\Lambda 0} + \Omega_{K0}\, x^{-2} + \Omega_{m0}\, x^{-3} + \Omega_{r0}\, x^{-4}}}\right]$$

Light travel time

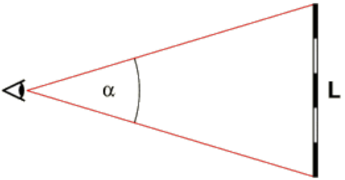
In astronomy, measuring distance by light travel time is common!

Determine travel time by using earlier expression relating $\mathrm{d}t$ and $\mathrm{d}x$ and integrating up:

$$t_0 - t(z) = \frac{1}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x\,\sqrt{\Omega_{\Lambda 0} + \Omega_{K0}\,x^{-2} + \Omega_{m0}\,x^{-3} + \Omega_{r0}\,x^{-4}}}.$$

Angular distance

Consider an object at redshift z with (proper) size L:



Under what angle will we see that object? Go back to FRW metric:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right].$$

We've seen how light with ds^2 travels in the radial direction. Consider two *light rays* reaching us with a (small) angular difference $\Delta\theta = \alpha$.

Angular distance

Now consider the time t_1 when the light was emitted. Use the metric and insert the angular difference $\Delta\theta$:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right].$$

$$\Rightarrow \mathrm{d}s = a(t_1)r_E\,\alpha = L.$$

Define **angular distance** analogously to classical geometry:

$$d_A(z) = \frac{L}{\alpha} = a(t_1)\,r(z) = \frac{a_0}{1+z}\,r(z) = \frac{d_{\mathrm{now}}(z)}{1+z}$$

(cf. explicit formula for d_{now} calculated earlier).

Classical luminosity distance

Absolute luminosity L is total energy emitted by an object per second.

Apparent luminosity (energy flux) f is the energy received per second per unit area.

For isotropic brightness: total energy passes through sphere with radius r, so

$$f = \frac{L}{4\pi r^2}.$$

If L is the same for each object in a certain class, or can be determined from observations, we have a **standard candle**.

FRW luminosity distance

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right].$$

Corrections to classical derivation for light emitted at time t_1 by object at redshift z:

- Energy emitted at time t_1 has spread out on sphere with proper area $4\pi r_1(z)^2 a^2(t_0)$ (use symmetry between the object's and our own position, $r_1(z) = r(z)$)
- Photons arrive at a lower rate, given by redshift factor $a(t_1)/a_0 = 1/(1+z)$
- Photon energy is $E = h\nu$; redshift reduces energy by $1/(1+z)$

FRW luminosity distance

Result:

$$f = \frac{L}{4\pi r(z)^2 a_0^2 (1+z)^2}$$

Define luminosity distance by

$$f = \frac{L}{4\pi d_L(z)^2},$$

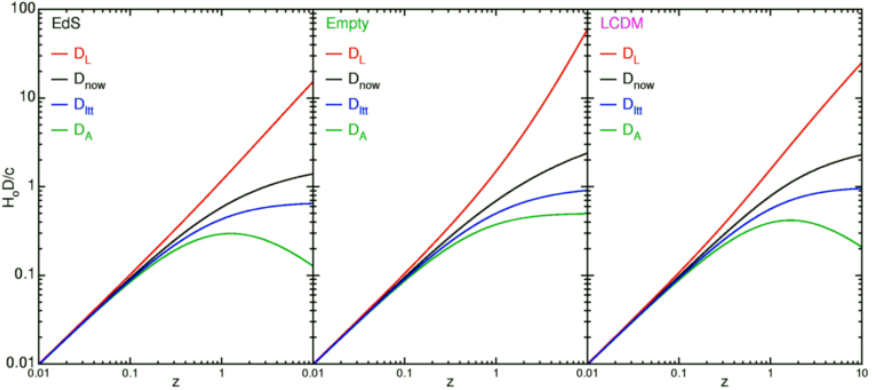
so

$$d_L(z) = a_0 r(z) \cdot (1+z) = d_{now}(z)(1+z) = d_A(z) \cdot (1+z)^2.$$

Different notions of distance

1. **redshift** z — for monotonously expanding universe, good distance measure; model-independent, can be measured directly
2. **proper distance** d_{now} — instantanous (w.r.t. cosmic time) distance
3. **co-moving distance** r — coordinate distance, useful for tagging
4. **light-travel time** — the original light year
5. **angular distance** d_A — for observations of standard rulers
6. **luminosity distance** d_L — for observations of standard candles

Different notions of distance



From: Ned Wright's cosmology tutorial. EdS = Einstein–de Sitter universe, $K = 0, \Omega_{\Lambda 0} = 0$.

Horizons

Causal structure of spacetime: Which parts are accessible? Which are inaccessible?

Most prominent example: Black holes with their event horizon — what's behind the horizon cannot communicate with the outside.

Two varieties: **particle horizon** and **event horizon**.

Particle horizons

In a universe with finite age, the *observable universe* is finite, as well.

Re-writing the FLRW metric once more, using $\mathrm{d}s^2 = 0$ to describe light reaching us at the present time, t_0, from some distance r. Light with r_{max} has been travelling since the big bang ($t = 0$):

$$\int_0^{t_0} \frac{c\,\mathrm{d}t'}{a(t')} = \int_0^{r_{\mathrm{max}}} \frac{\mathrm{d}r'}{\sqrt{1 - Kr'^2}}.$$

But we do not even need to solve for r_{max}, since what we're really interested in is the proper distance:

$$d_{\mathrm{particle}}(t_0) = a_0 \int_0^{r_{\mathrm{max}}} \frac{\mathrm{d}r'}{\sqrt{1 - Kr'^2}} = a_0 \int_0^{t_0} \frac{c\,\mathrm{d}t'}{a(t')}.$$

Particle horizons

$$d_{\text{particle}}(t_0) = a_0 c \int_0^{t_0} \frac{\mathrm{d}t'}{a(t')}$$

One possible definition for the observable universe!

Event horizons

Which events happening at present will we see? Which not?

Same basic derivation from FLRW metric:

$$\int_{t_0}^{t_{max}} \frac{c\,\mathrm{d}t'}{a(t')} = \int_0^{r_{max}(t_0)} \frac{\mathrm{d}r'}{\sqrt{1-Kr'^2}}.$$

t_{max} is infinite for infinitely expanding universes, finite for re-collapsing ones. We're again interested in proper distances:

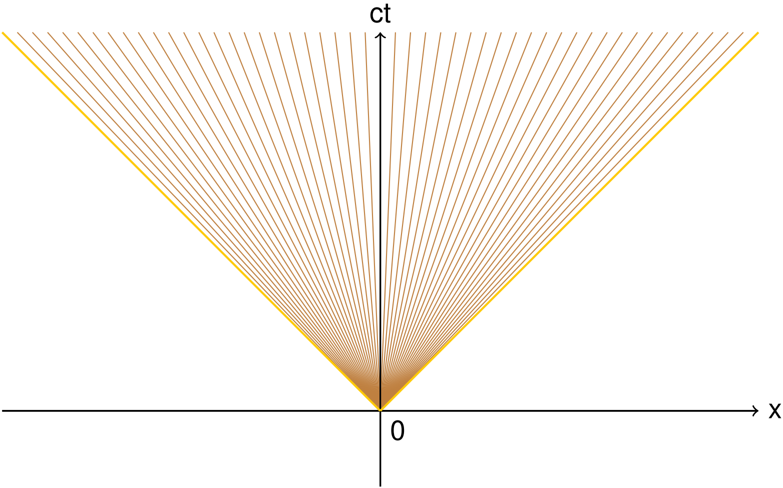
$$d_{event}(t_0) = a_0 c \int_0^{r_{max}(t_0)} \frac{\mathrm{d}r'}{\sqrt{1-Kr'^2}} = a_0 c \int_{t_0}^{t_{max}} \frac{\mathrm{d}t'}{a(t')}.$$

The Milne universe: The kinematics of expansion

Edmund Arthur Milne (1932) — meant as an alternative to gr; modern view: instructive idealized case. FLRW universe with $\rho = 0$ and $p = 0$ — no dynamics, only kinematics.

Imagine an explosion at the origin of Minkowski space, sending particles in all directions. These particles have all possible speeds, right up to as close to c as you can get.
What does an observer in such a cosmos see and measure?

Milne universe: Space-time diagram



Proper times and particle density in the Milne universe

In our inertial frame, each particle is characterized by a constant speed v, moving radially with $r = vt$.

Let each particle carry a clock showing its proper time τ, set to zero at the explosion event in the origin.

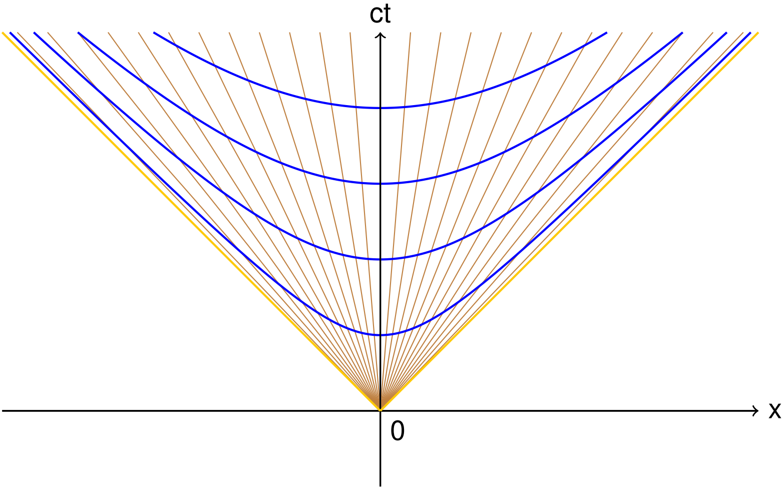
Then

$$\tau^2 = t^2 - r^2/c^2 = t^2/\gamma^2(r/t)$$

so surfaces of equal τ are hyperbolas.

Note analogy with definition of cosmic time – via proper time of substratum particles!

Cosmic time in the Milne universe



Spatial metric on surfaces τ =const.?

With τ our cosmic time, how do we measure distances at one instant in time, $\tau = \tau_0$ =const.?

For this case, from $\tau_0^2 = t^2 - r^2/c^2$

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 + (r/c\tau_0)^2}.$$

Introduce a "proper length" coordinate χ (proper distance from the reference particle, at rest at $r = 0$) as

$$\chi = \int_0^\chi \mathrm{d}s = \int_0^r \frac{\mathrm{d}r}{\sqrt{1 + (r/c\tau_0)^2}} = c\tau_0 \ \sinh^{-1}(r/c\tau_0).$$

Cosmological coordinates for the Milne universe

New coordinates τ, χ related to t, r as

$$
\begin{aligned}
t &= \tau\cosh(\chi/c\tau)\\
r &= c\tau\sinh(\chi/c\tau)
\end{aligned}
$$

(θ, ϕ kept the same).
Induced metric from

$$
\begin{aligned}
\mathrm{d}t &= \mathrm{d}\tau\ [\cosh(\chi/c\tau) - \chi/c\tau\ \sinh(\chi/c\tau)] + \sinh(\chi/c\tau)\ \mathrm{d}\chi\\
\mathrm{d}r &= c\mathrm{d}\tau\ [\sinh(\chi/c\tau) - \chi/c\tau\ \cosh(\chi/c\tau)] + \cosh(\chi/c\tau)\ \mathrm{d}\chi
\end{aligned}
$$

$$
\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2\left(1-\frac{\chi^2}{(c\tau)^2}\right) - 2\frac{\chi}{c\tau}\,\mathrm{d}\tau\,\mathrm{d}\chi + \mathrm{d}\chi^2 + (c\tau)^2\sinh(\chi/c\tau)\mathrm{d}\Omega
$$

Cosmological coordinates for the Milne universe

This is not quite where we want to be — we want co-moving coordinates, so for particles with constant spatial coordinates, $\mathrm{d}\tau$ is proper time. Ansatz:

$$\chi = a(\tau)\xi \quad \Rightarrow \quad \mathrm{d}\chi = \dot{a}\,\xi\,\mathrm{d}\tau + a\,\mathrm{d}\xi.$$

Once more, substitute in the metric:

$$\begin{aligned}\mathrm{d}s^2 \quad = \quad & -c^2\mathrm{d}\tau^2\left(1 - \frac{a\xi^2}{c^2\tau}\left[\dot{a} - \frac{a}{\tau}\right]\right) + 2\mathrm{d}t\,\mathrm{d}\xi\,\xi a\left[\dot{a} - \frac{a}{\tau}\right] \\ & +a^2\mathrm{d}\xi^2 + (c\tau)^2\sinh(a\xi/c\tau)\mathrm{d}\Omega\end{aligned}$$

This takes on the desired form for $\dot{a}a/\tau$, or $a(\tau) \propto \tau$. For simplest form, choose $a(\tau) = c\tau$.

Cosmological coordinates for the Milne universe

Finally, in coordinates τ, ξ, θ, ϕ:

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 + a^2(\tau)\left[\mathrm{d}\xi^2 + \sinh^2(\xi)\mathrm{d}\Omega\right]$$

with $a(\tau) = \tau$.

In comparison with the hyperbolic metric we derived earlier, this is FLRW with $K = -1$, a hyperbolic homogeneous space.

This also shows that space geometry is indeed homogeneous: With Lorentz transformations, we can shift the "center of the universe" to any location, and the result will be the same!

Density of the Milne universe

So far, we haven't specified the density of the Milne universe — which should be homogeneous in cosmological coordinates!

Take some reference time τ_0, and assume that in our co-moving coordinate system, at that time, a sphere of volume V_0 contains N particles, making for number density $n_0 = N/V_0$. At some later time τ, the same N particles will now be contained in a volume of

$$V(\tau) = \left(\frac{a(\tau)}{a(\tau_0)}\right)^3 V_0 = \tau^3 \frac{V_0}{\tau_0^3}.$$

Thus (proper) particle number density goes as

$$n(\tau) = \tau^{-3} n_0\, \tau_0^3.$$

Density of the Milne universe

Now, go back to r, t coordinates. From this point of view,

$$n(\tau) = \tau^{-3} n_0\, \tau_0^3$$

is a proper particle density; in our external reference frame, it is Lorentz-contracted in the direction of motion, giving a number density $n_{ext} = n(\tau)\gamma$. But γ can be written as $t = \gamma\tau$, so

$$n_{ext}(t, r) = n(\tau)\gamma = \frac{n(\tau)}{\tau} t = \frac{n_0/\tau_0^3}{\tau^4} t = n_0/\tau_0^3 \frac{t}{(t^2 - r^2/c^2)^2}.$$

This is clearly singular at the light-cone – which closes off the universe as soon as we switch on gravity!

Superluminal galaxies?

Distance $\chi = a(\tau)\xi = c\tau\xi$ from galaxy at co-moving coordinate value ξ changes as

$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = c\xi.$$

For $\xi > 1$, this is clearly "superluminal". Yet we are safely in the framework of special relativity – this "speed" is a mere coordinate artefact!

The cosmological redshift in the Milne model

What about light reaching us (at the "center particle") at time t_0, emitted at distance r_e at time t_e?

Relativistic Doppler shift says

$$z + 1 = \sqrt{\frac{1 + r_e/ct_e}{1 - r_e/ct_e}}.$$

Cosmological redshift says

$$z + 1 = \frac{a(\tau_0)}{a(\tau_e)} = \frac{\tau_0}{\tau_e}.$$

The cosmological redshift in the Milne model

But by $t = \tau\cosh(\chi/c\tau),\ r = c\tau\sinh(\chi/c\tau),\ \tau^2 = t^2 - (r/c)^2$, and since

$$t_0 = t_e + r_e/c:$$

$$\frac{\tau_0}{\tau_e} = \frac{t_e + r_e/c}{\sqrt{t_e^2 - r_e^{/}c^2}} = \sqrt{\frac{(t_e + r_e/c)^2}{(t_e - r_e/c)(t_e + r_e/c)}} = \sqrt{\frac{1 + r_e/ct_e}{1 - r_e/ct_e}}$$

The special-relativistic Doppler shift gives the correct cosmological redshift!

This is true more generally, as long as gr's rules for comparing velocities are taken into account properly, cf. Bunn & Hogg arXiv:0808.1081

Local effects of expansion?

Does expansion have an effect locally? Do atoms, planetary orbits, galaxies expand? cf. Giulini, arXiv:1306.0374v1

Overall: Average density means no net force on, say, galaxies ⇒ expansion on largest scales. But what about bound systems?

Pseudo-Newtonian picture: The different inertial frames are "moving away" from each other by the expansion,

$$\ddot{\vec{x}} = \frac{\ddot{a}}{a}\vec{x} \approx -q_0 H_0^2\,\vec{x}$$

gives additional term in Newton's equations,

$$m(\ddot{x} - \frac{\ddot{a}}{a}\vec{x}) = \vec{F}.$$

Only $\ddot{a}$ matters, not $\dot{a}$! Not some "friction force" pulling everything along with expansion!

Expansion and the Coulomb potential 1/3

Setting up a modified Coulomb potential (electromagnetism, gravity): Energy and angular momentum

$$\frac{1}{2}\dot{r}^2 + U(r) = E, \quad r^2\dot{\phi} = L,$$

with the effective potential

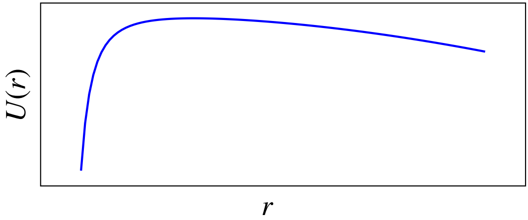
$$U(r) = \frac{L^2}{2r^2} - \frac{C}{r} + \frac{1}{2}Ar^2,$$

where

$$C = \begin{cases} GM & \text{gravitational field} \\ \frac{Qe}{4\pi\epsilon_0 m} & \text{electric field} \end{cases}$$

and $A = -q_0 H_0$.

Expansion and the Coulomb potential 2/3



Critical radius at

$$r_c = \sqrt[3]{\frac{C}{A}}.$$

Amounts to

$$r_c = \begin{cases} \left(\frac{M}{M_\odot}\right)^{1/3} 108\text{pc} & \text{gravity} \\ \left(\frac{Q}{e}\right)^{1/3} 30\text{AU} & \text{electrostatic} \end{cases}$$

Expansion and the Coulomb potential 3/3

$$r_c = \begin{cases} \left(\frac{M}{M_\odot}\right)^{1/3} 108\text{pc} & \text{gravity} \\ \left(\frac{Q}{e}\right)^{1/3} 30\text{AU} & \text{electrostatic} \end{cases}$$

means that:

- for a hydrogen atom, instead of the Sun, the electron would have to be near Pluto
- for the Sun, planets would need to be far beyond the neighbouring stars
- for a galaxy at $10^{12}\,M_\odot$, next galaxy beyond 1 Mpc

Recall $q_0 = \frac{4\pi G}{3}(\rho_0 + 3p_0)$. — for ordinary Dark Energy, density/pressure are constant. If those evolve, as in some *quintessence* models, there could be a “big rip”!

The tethered galaxy problem 1/2

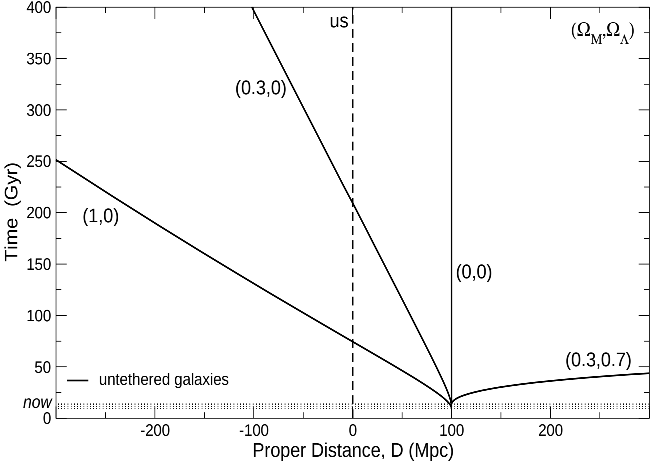
Assume a galaxy that, in cosmological coordinates, is momentarily at rest (that is, its proper distance doesn't change to first order): How will that galaxy move?

Naïve "Space is expanding" view would expect: Galaxy slowly begins to move away from us.

Instead: H_0 doesn't have any influence at all; purely dynamical effects!

In particular, without dark energy, galaxy moves *towards* us before sorting itself kinematically into the Hubble flow.

The tethered galaxy problem 2/2



Davis et al. arXiv:astro-ph/0104349

So what is the expansion of space?

As Milne, tethered galaxy, bound systems would suggest:

- First-order expansion rate H_0 is kinematical – depends on initial conditions of substrate particles
- Smaller, dynamical effects $(\Omega_{m0}, \Omega_{\Lambda 0}, \ldots)$ affect all particles present
- Initial conditions are important!
- Analogies like rubber band space, space as a viscuous fluid, space being created between galaxies are misleading
- Yes, the cosmological redshift can be understood as a relativistic Doppler shift
- No, there are no superluminal galaxies if you take proper care not to over-interpret coordinates

. . . but this question still leads to amazingly heated discussions!

Literature: Cosmology

Little, Andrew J.: *An Introduction to Modern Cosmology*. Wiley 2003 [brief and basic]

Lambourne, Robert J.: *Relativity, Gravitation and Cosmology.* Cambridge University Press 2010

Dodelson, Scott: *Modern Cosmology*. Academic Press 2003. [more advanced]

Weinberg, Steven: *Cosmology*. Oxford University Press 2008 [advanced]

Script by Matthias Bartelmann http://www.ita.uni-heidelberg.de/research/bartelmann/files/cosmology.pdf