Introduction, General Relativity, FLRW Spacetime

Cosmology Block Course 2014

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Contents

Building the biggest model of all

- Physicists *always* build models
- Model: simplified representations of some part of reality, capturing essential aspects
- Model simplifications: Spatial restriction; sub-structure neglected (e.g. continuum mechanics)
- How can one model the whole universe?
- Apparently, that depends on the universe: some are modelable, some not.
- . . . so what are the properties of our universe?

Free lines-of-sight!

Imagine a:

- planet in a dust envelope
- Solar system in a dust envelope
- Solar system in a dense globular cluster
- ...

$\Rightarrow$ we can hope to make statements about the universe as a whole because we can see to great distances!

Free lines-of-sight!

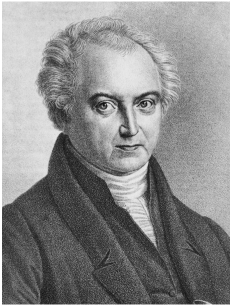
Hubble Deep Field

Lookback time
> 12 Gyr

Credit: R. Williams (STScI), the Hubble Deep Field Team and NASA

Olbers' paradox (1823)

Heinrich Wilhelm Matthias Olbers (1758-1840):



The universe cannot be infinite and stationary

If it were: Every line-of-sight would end in a star; constant surface as luminosity goes with $1/r^2$ but angular area with r^2.

(Dust/absorption? If stationary, thermal equilibrium would give dust the same surface brightness!)

Large-scale homogeneity/isotropy vs. structure

Stellar densities $\sim 10^3$ kg/m^3 on scales of 10^6 m

vs.

Interstellar medium, density $\sim 10^{-21}$ kg/m^3,
average interstellar distances $10^{16} \ldots 10^{17}$ m

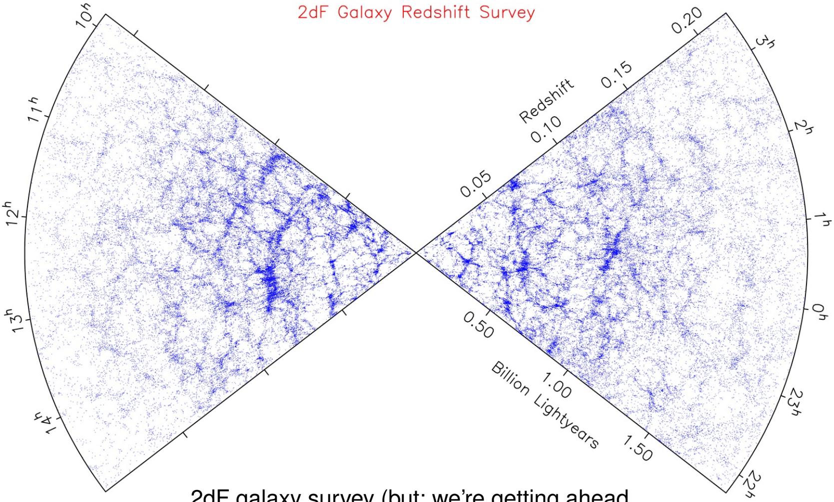
Large-scale homogeneity/isotropy vs. structure

Galactic densities (including DM) $\sim 10^{-24}$ kg/m^3
on scales of 10^{22} m (including halo)
(after arXiv:0801.1232v5 p. 16 - virial radius)

vs.

Intergalactic density (gas + DM) $\sim 10^{-27}$ kg/m^3,
intergalactic distances $10^{22} \ldots 10^{23}$ m

Large-scale homogeneity/isotropy vs. structure



2dF galaxy survey (but: we're getting ahead of ourselves; distances measured via redshift)

Systematic redshift-distance relations

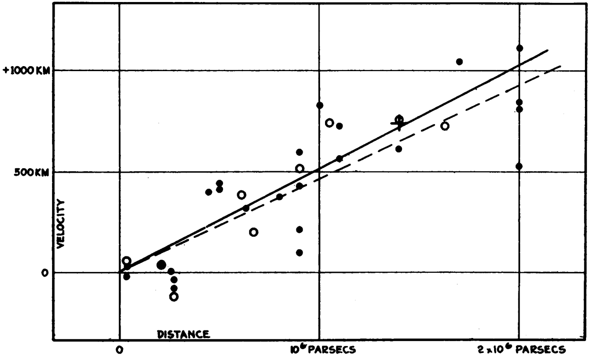
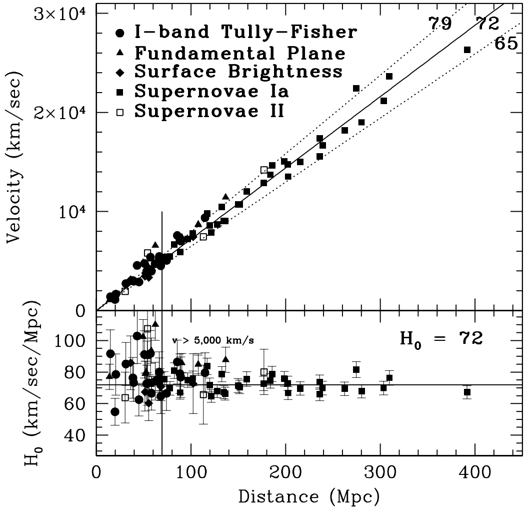


FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.

Hubble 1929: “A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae” in PNAS 15(3), S. 168ff.

HST Key Project results



From Freedman 2001 et al. (HST Key Project)

Putting it all (almost) together

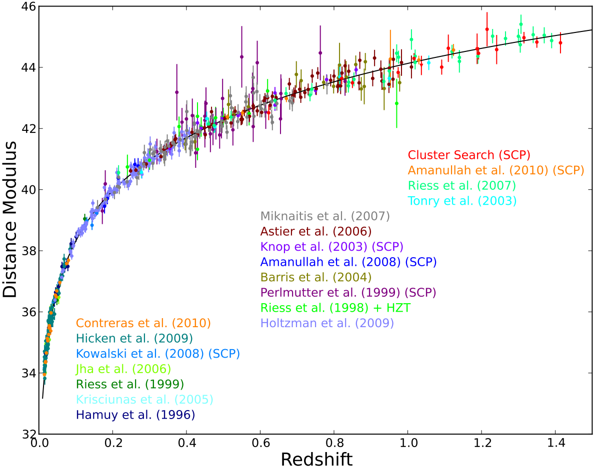
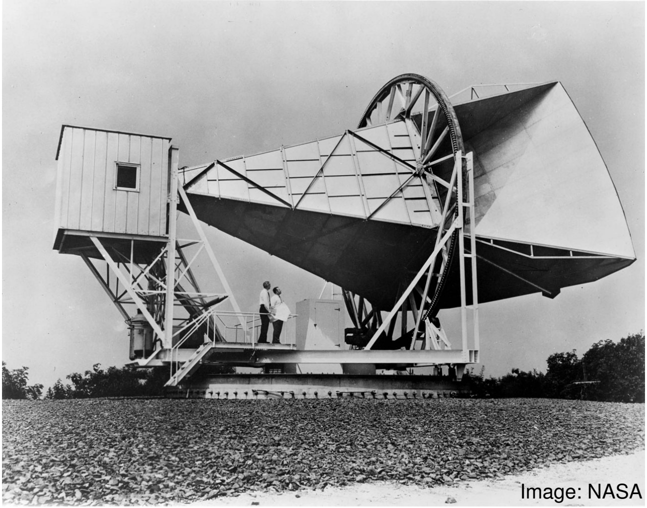
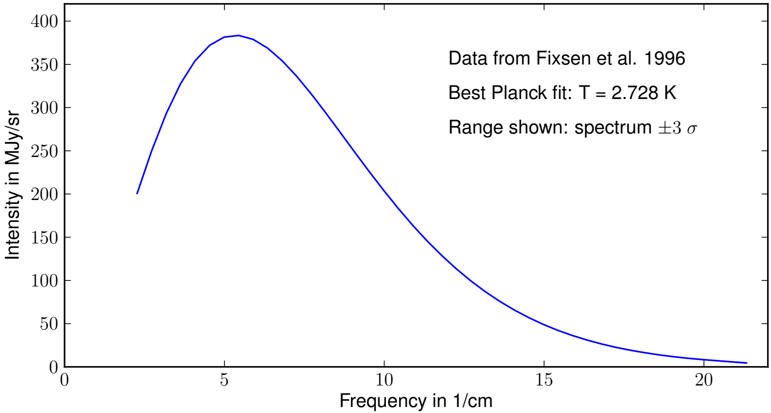


Image: Suzuki et al. 2011

Cosmic microwave background: Penzias & Wilson



Precision CMB: COBE-FIRAS (Mather et al.)



Data from Fixsen et al. 1996 via http://lambda.gsfc.nasa.gov

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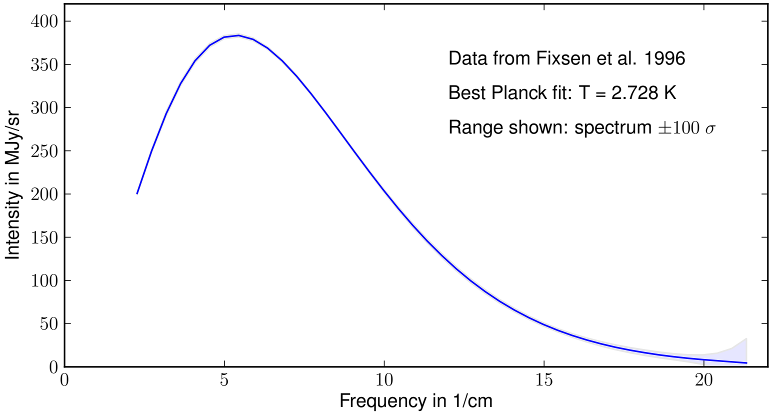

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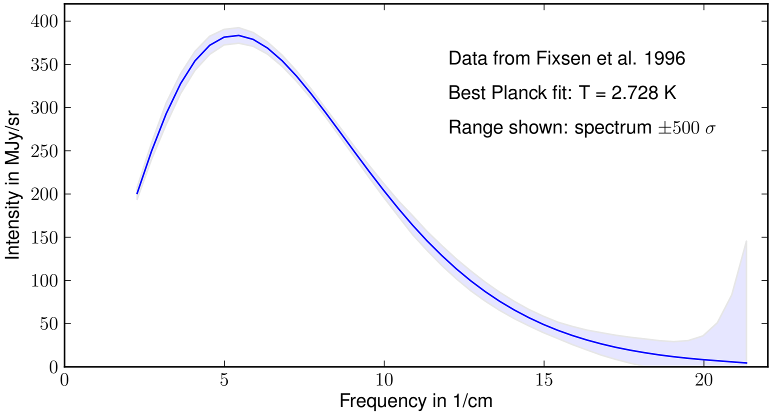

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CMB inhomogeneities

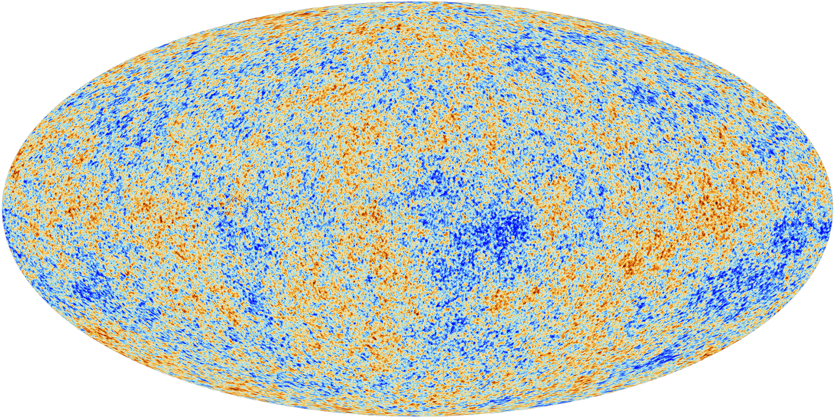


Image: ESA/Planck Collaboration

CMB inhomogeneities: Power spectrum

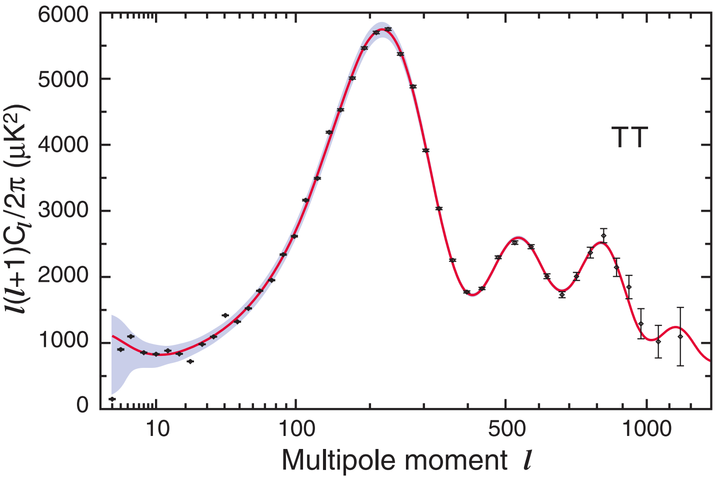


Fig. 32 in Bennett et al. 2013

Matter content of the universe

- Stars: Easy to detect! (Extinction maps needed, though)
- Dust within our galaxy: IR observations
- Atomic hydrogen: 21 cm line, absorption lines
- Molecules: IR, radio
- Very distant warm plasma: Hard to detect!

More general mass measurements: Use gravitational probes (e.g. satellite galaxies orbiting a galaxy) as tracers.
Virial measurements: Dispersion σ related to attracting mass by

$$\sigma^2 \sim \frac{GM}{R}.$$

Matter content: Overall density

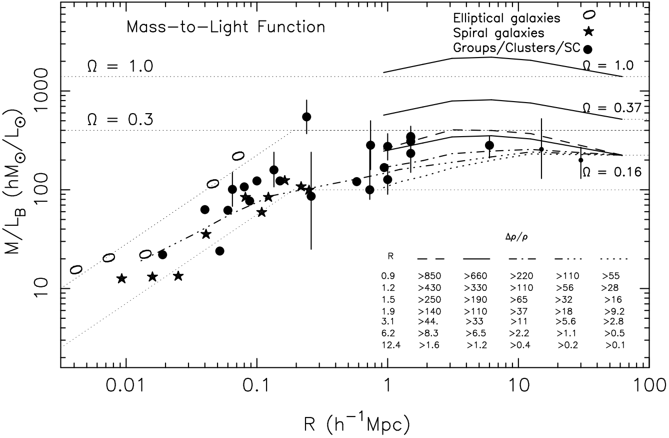
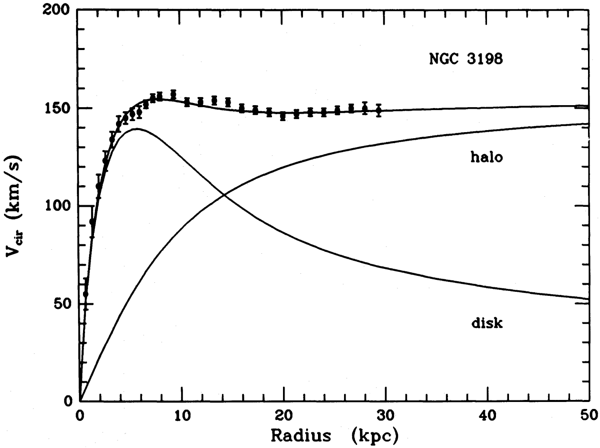


Fig. 2 in Bahcall et al. 2000, arXiv:astro-ph/0002310

where $\Omega \approx \rho/(10^{-26}\,\text{kg/m}^3)$

Dark matter

Deviation from Kepler potential as generated by visible contributions to mass (here van Albada et al. 1985):



Dark matter

- no electromagnetic interaction, just gravitational
- first postulated by Fritz Zwicky to explain motion within galaxy clusters (virial theorem)
- direct detection experiments: inconclusive and, currently, somewhat contradictory
- WIMPs: particles based on supersymmetric extensions? $\Rightarrow LHC$
- several sort-of-independent types of evidence:
  - Galaxy rotation curves
  - Dynamics of galaxy clusters
  - Gravitational lensing (including Bullet cluster)
  - Cosmological (later): Fluctuations in primordial plasma
- (or alternatively: modified dynamics, i.e. MOND?)

Matter content of the universe

$$\Omega_m = \left\{ \begin{array}{lcr} \Omega_b & = & 4.9\% \\ \Omega_d & = & 26.8\% \end{array} \right\} = 31.7\%$$

$$\Omega_r = 0.005\%$$

$$\Omega_\Lambda = 68.3\%$$

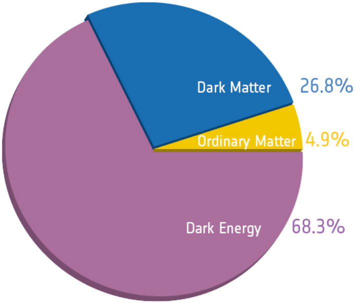


Image credit: ESA/Planck Collaboration

Wobei Ω_b = ordinary, baryonic matter (protons, neutrons, ...)
Ω_d = dark matter (no interaction with light)
Ω_Λ = dark energy (whatever that is, but it accelerates the expansion)

Age determinations

Trivially, nothing in the universe can be older than the universe itself.

(There was a time when that appeared to be a problem!)

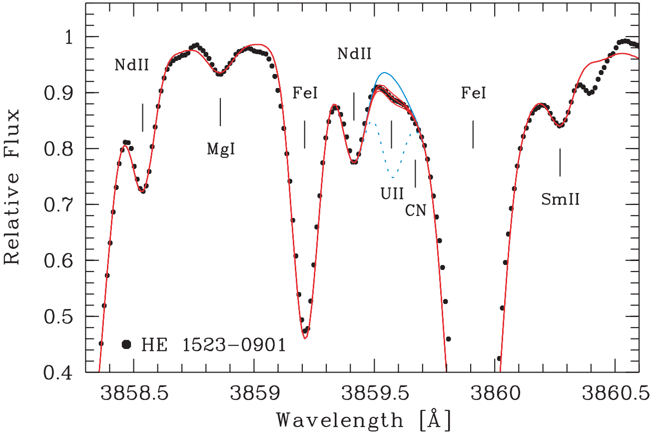
First possibility: Radioactive dating. Some half-life values:

^{235}U	$7 \cdot 10^8$ a
^{232}Th	$1.4 \cdot 10^{10}$ a

$\Rightarrow$ Heavy elements formed in the r-process (rapid addition of neutrons) in core-collapse supernovae (some modelling involved!)

HE 1523-0903

Example for very old, metal-poor star (Frebel, Christlieb et al. 2007): U- and Th- dated to 13.2 Gyr!

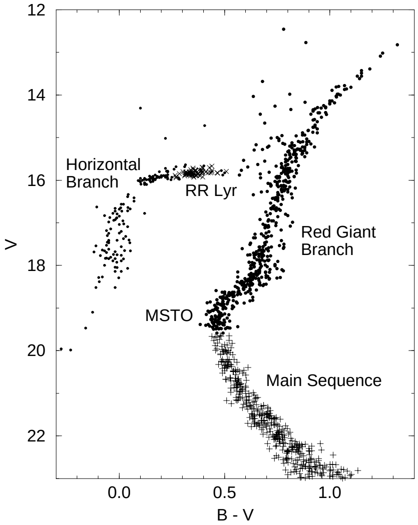


Stellar ages

Model for stellar evolution: stars move in the *Hertzsprung-Russell diagram* (color-magnitude diagram) as they evolve.

Lifetime $\tau \sim L^{-2/3}$, $L \sim M^3$ and $\tau \sim T^{-1}$.

Oldest globular clusters give 13.2 ± 2 Gyr (Carretta et al. 2000).



Other relevant observations

. . . more specialized, and directly in response to cosmological models:

- Number counts by distance (to counter Steady State theory)
- Power spectrum of galaxy distribution by distance: Baryonic Acoustic Oscillations
- Tolman's surface brightness test (Lubin & Sandage 2001)
- SN light curve time dilation (Leibundgut et al. 1996)

Cosmological model-building

Simplest cosmological models:
Homogeneous and isotropic universes

Alternative definition:
Copernican principle/Cosmological principle: We occupy no special location in the universe.

Universe filled with a fluid, the “cosmic substrate” — at early times, primordial plasma; at later times, with “galaxy dust”

Overview of cosmological modelling

Homogeneous models

General relativity
FLRW spacetimes
H_0 kinematics
$\Omega_m, \Omega_\Lambda, \Omega_b, \Omega_r$ dynamics

Early, hot universe

Thermodynamics/Statistics
Particle, nuclear, atomic ph.
η baryon-photon ratio
inflaton properties

Inhomogeneities

Newtonian perturbations
Newtonian numerics
Raytracing
power spectrum
scalar vs. tensor
reionization time

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General relativity (1915)

- Einstein's theory linking gravity with space-time geometry
- Connection made by **Einstein (field) equations**
- generalization of special relativity
- geometry in general non-Euclidean (curved)
- basic descriptor of space-time geometry: **metric**
- sources of gravity: **mass, energy, pressure**

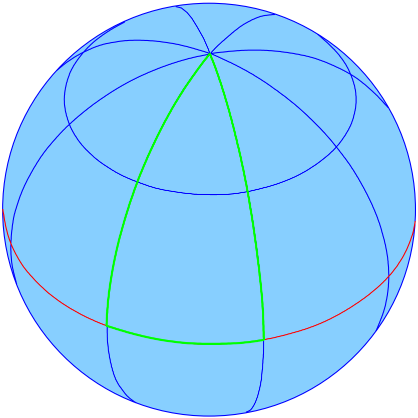
For cosmology:
We need to **understand space-time geometry** (necessary to understand light propagation, horizons, age of universe, distances)
We will **take as given** what gr says about the dynamics of homogeneous/isotropic universes

General relativity vs. curved surfaces

4D space-time	⇔	2D (curved) surface
Particle worldline	⇔	curve on surface
Free-fall worldline	⇔	straightest-possible lines on surface (geodesics)
Equivalence principle: in free fall, physics = special relativity	⇔	on infinitesimal scales, curved surface looks flat
deviation from flatness: curvature tensor(s)	⇔	deviation from plane: curvature radii

Geometry is encoded in a mathematical object: the **metric**.
We need to know how to interpret a metric!

A simple curved surface: the sphere



[more info on the blackboard]

Introducing general coordinates

The three-fold use of coordinates:

- Labels to identify points
- Encode closeness (topological space)
- Encode distances (space with metric, e.g. $l = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$)

As we generalize from simple, Euclidean space, we will have to look at these roles in turn!

Coordinates on a wavy surface

Let's begin in two dimensions: with a smooth, but wavy, hilly surface ("Buckelpiste"):

Image: Andreas Hallerbach under CC-BY-NC-ND 2.0

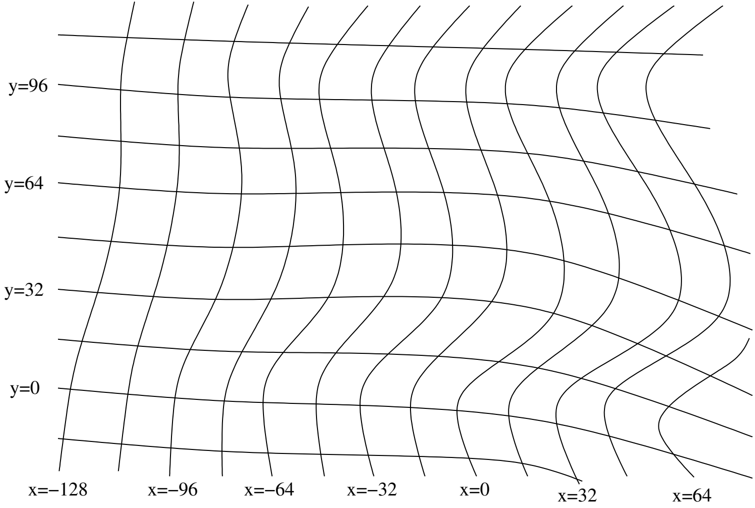
Coordinates on a wavy surface

Even better: Imagine that the surface is pure, smooth rock.

Now, put coordinate lines on it. (Purpose, for a start: Identifying different points.)

The lines are going to be curvy and wavy.

Coordinates on a wavy surface



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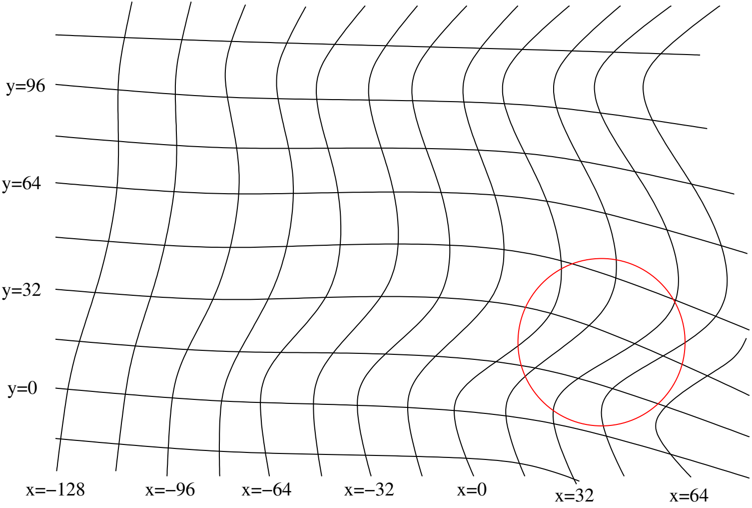

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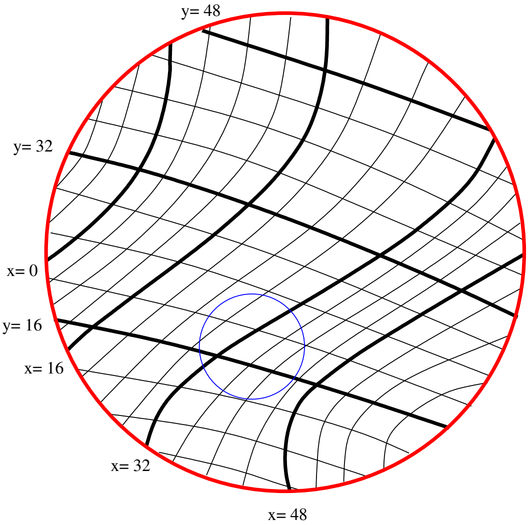

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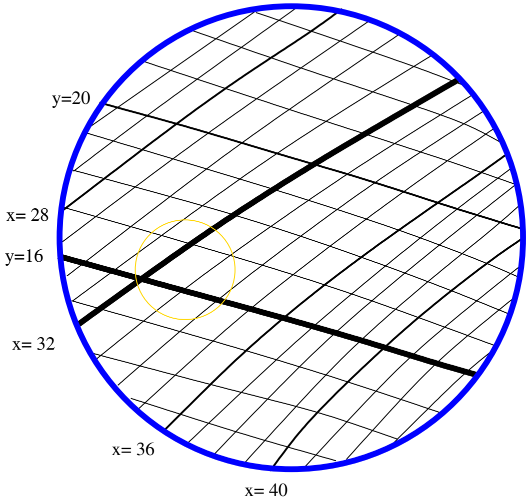

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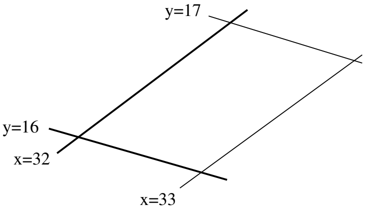
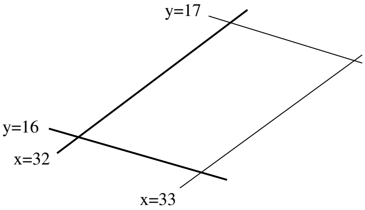


This if fairly simple - a parallelogram!

Coordinates on a wavy surface



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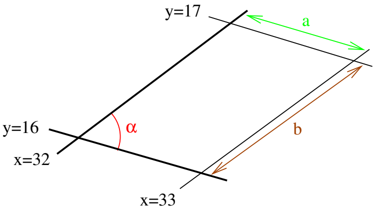
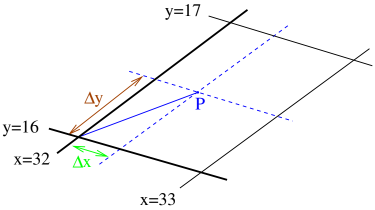


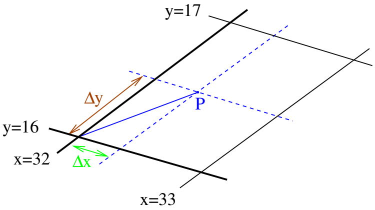
Assume an isometric view (straight down onto the plane): read off 3 parameters!

Coordinates on a wavy surface



What's the length of the blue line between $(32, 16)$ and P?

Coordinates on a wavy surface



$\vec{P} = (b\,\Delta y)\,\vec{u}_y + (a\,\Delta x)\,\vec{u}_x$ where $\vec{u}_x \cdot \vec{u}_y = \cos\alpha$ means that

$$|\vec{P}|^2 = a^2\,\Delta x^2 + 2ab\,\cos\alpha\,\Delta x\Delta y + b^2\,\Delta y^2.$$

With this modification, our coordinates can be used to measure lengths!

Defining the metric 1/2

$$|\vec{P}|^2 = a^2\,\Delta x^2 + 2ab\,\cos\alpha\,\Delta x\Delta y + b^2\,\Delta y^2.$$

This was really an infinitesimal argument (lengths in the neighbourhood of P):

$$\mathrm{d}s^2 = a^2\,\mathrm{d}x^2 + 2ab\,\cos\alpha\,\mathrm{d}x\mathrm{d}y + b^2\,\mathrm{d}y^2.$$

The coefficients will vary from location to location:

$$\mathrm{d}s^2 = a(x,y)^2\,\mathrm{d}x^2 + 2a(x,y)b(x,y)\,\cos[\alpha(x,y)]\,\mathrm{d}x\mathrm{d}y + b(x,y)^2\,\mathrm{d}y^2.$$

If we know all the coefficients, we can reconstruct the geometry of the whole surface (except for embedding properties): The coefficients, all taken together, form the **metric**

Defining the metric 2/2

Metric (working definition): A set of (position-dependent) coefficients that allow one to compute lengths from infinitesimal coordinate differences.

2D example:

$$\begin{aligned} \mathrm{d}s^2 &= a^2\,\mathrm{d}x^2 + 2ab\,\cos\alpha\,\mathrm{d}x\mathrm{d}y + b^2\,\mathrm{d}y^2 \\ &= (\mathrm{d}x, \mathrm{d}y)\begin{pmatrix} a & ab\cos\alpha \\ ab\cos\alpha & b \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} \end{aligned}$$

The metric can be written as a *symmetric matrix*, or a *quadratic form*. Taking coordinate transformations into account, it behaves like what is called a *(symmetric, second-rank) tensor*.

Writing the metric

Usual symbol for the metric: g

Line-element notation in D dimensions:

$$\mathrm{d}s^2 = \sum_{i,j=1}^{D} g_{ij}(x)\mathrm{d}x_i\mathrm{d}x_j$$

with g_{ij} the *metric coefficients*.

In our simple example: $g_{11} = a$, $g_{12} = g_{21} = ab\cos\alpha$, $g_{22} = b$.

Examples for metrics: Euclidean

Cartesian coordinates in 3D Euclidean space

Pythagoras says:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Examples for metrics: Spherical

Spherical coordinates in Euclidean space:

$$
\begin{aligned}
x &= r\sin(\theta)\cos(\phi) \\
y &= r\sin(\theta)\sin(\phi) \\
z &= r\cos(\theta)
\end{aligned}
$$

Examples for metrics: Spherical

$$\begin{aligned}
\mathrm{d}x &= \mathrm{d}r\ \sin(\theta)\cos(\phi) + r(\cos(\theta)\ \mathrm{d}\theta\ \cos(\phi) - \sin(\theta)\sin(\phi)\ \mathrm{d}\phi) \\
\mathrm{d}y &= \mathrm{d}r\ \sin(\theta)\sin(\phi) + r(\cos(\theta)\ \mathrm{d}\theta\ \sin(\phi) + \sin(\theta)\cos(\phi)\ \mathrm{d}\phi) \\
\mathrm{d}z &= \mathrm{d}r\ \cos(\theta) - r\sin(\theta)\ \mathrm{d}\theta.
\end{aligned}$$

Line element is:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2).$$

Tricky: visual inspection of metric doesn’t tell you: unusual coordinates or curved surface?

Examples for metrics: Embedded spherical surface

Line element is:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2).$$

Restrict to $\mathrm{d}r = 0$ — use θ, ϕ as coordinates on the surface (think: latitude, longitude). This gives (induced) metric on the surface of a sphere:

$$\mathrm{d}s^2 = r^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2).$$

with some r =const. the radius of the sphere — which can be used to calculate arc lengths etc.!

Special relativity: Minkowski metric

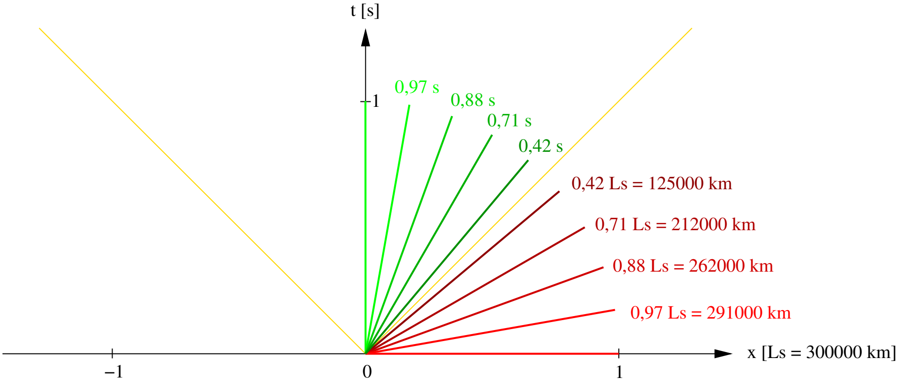
One can define a metric in special relativity, but it doesn't look like the ones we've encountered. This is the *Minkowski metric*:

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = \mathrm{d}\vec{x}^2 - c^2\mathrm{d}t^2.$$

This is invariant under Lorentz transformations!

But what does it mean?

The meaning of the SR metric



$$ds^2 = -c^2 d\tau^2 = d\vec{x}^2 - c^2 dt^2.$$

The meaning of the SR metric

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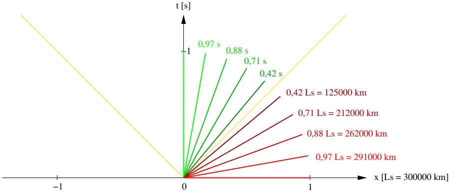


- **timelike**, $ds^2 < 0$: possible worldlines of $(m > 0)$ particles
- **lightlike**, $ds^2 = 0$: light-cone
- **spacelike**, $ds^2 > 0$: possible spatial distance

Preparation for large-scale cosmic geometry

Natural coordinates for a homogeneous universe: 3D space is homogeneous, as well.
Rigorous route: Killing vectors & form invariance, cf. sec. 13 in Weinberg (1972)

Simpler question: What can we think of?

- Euclidean 3D space
- Embeddings, as in our derivation of the metric of the 2D spherical surface

Choice of spatial metric: Euclidean

Euclidean space:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \equiv \mathrm{d}\vec{x}^2.$$

$$\mathrm{d}s^2 = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \mathrm{d}\vec{x}^T \cdot \mathrm{d}\vec{x}$$

...this is invariant under translations, since $\mathrm{d}(\vec{x}+\vec{a}) = \mathrm{d}\vec{x}$ and under rotation, since $\vec{x} \mapsto M\vec{x}$ with $M \in SO(3)$ means

$$\mathrm{d}(M\vec{x})^T \cdot \mathrm{d}(M\vec{x}) = \mathrm{d}\vec{x}^T \cdot M^T \cdot M \cdot \mathrm{d}\vec{x} = \mathrm{d}\vec{x}^T \cdot \mathrm{d}\vec{x}.$$

Choice of spatial metric: Spherical

What other homogeneous, isotropic spaces are there?

Think spherical; a spherical surface S^{n-1} embedded in $\mathbb{R}^n$ is defined as the union of all points with n-dimensional coordinates x_i where

$$\sum_{i=1}^{n} x_i^2 = R^2$$

with R the radius of the sphere. Two-sphere S^2: ordinary spherical surface in space.

At least locally: Use $n-1$ of the coordinates as coordinates on the surface, $\vec{x}$; one coordinate as embedding coordinate, ξ, then

$$\mathrm{d}s^2 = \mathrm{d}\vec{x}^2 + \mathrm{d}\xi^2 \quad \text{where} \quad \xi^2 + \vec{x}^2 = R^2.$$

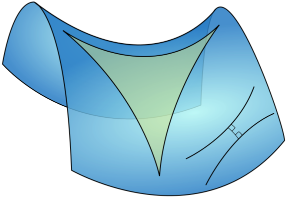
Choice of spatial metric: Spherical

$$\mathrm{d}s^2 = \mathrm{d}\vec{x}^2 + \mathrm{d}\xi^2 \quad \text{where} \quad \xi^2 + \vec{x}^2 = R^2$$

is invariant under rotations $M \in SO(4)$, which include homogeneity (any point can be rotated into any other point) and isotropy (any tangent vector can be rotated in any direction).

Easiest to see for $S^2 \in \mathbb{R}^3$: For each point P, one rotation (through embedding centerpoint and P) that will rotate space around P (isotropy), and two rotations that will shift the point into any given other point (homogeneity).

Choice of spatial metric: Hyperbolical



$$\mathrm{d}s^2 = \mathrm{d}\vec{x}^2 - \mathrm{d}\xi^2 \quad \text{where} \quad \xi^2 - \vec{x}^2 = R^2.$$

Higher-dimensional analogue of a saddle; invariant under $R \in SO(3,1)$.

This is the Lorentz group: SO(3) rotations (isotropy around each given point) and 3 Lorentz boosts that take the point into an arbitrary other point (homogeneity).

Unifying the spherical and hyperbolical spaces

Rescale $\vec{x} \mapsto \vec{x}/R$ and $\xi \mapsto \xi/R$:

$$\mathrm{d}s^2 = R^2 \left[\mathrm{d}\vec{x}^2 \pm \mathrm{d}\xi^2\right] \quad \text{where} \quad \xi^2 \pm \vec{x}^2 = 1.$$

From the constraint equation,

$$\mathrm{d}(\xi^2 \pm \vec{x}^2) = 0 = 2(\xi \mathrm{d}\xi \pm \vec{x} \cdot \mathrm{d}x)$$

relates the differentials. Substitute in metric to get unconstrained version:

$$\mathrm{d}s^2 = R^2 \left[\mathrm{d}\vec{x}^2 \pm \frac{(\vec{x} \cdot \mathrm{d}\vec{x})^2}{1 \mp \vec{x}^2}.\right]$$

Unifying the spherical and hyperbolical spaces

Introduce parameter $K = +1, 0, -1$ to write all three metrics in the same form:

$$\mathrm{d}s^2 = R^2 \left[\mathrm{d}\vec{x}^2 + K \frac{(\vec{x} \cdot \mathrm{d}\vec{x})^2}{1 - K\vec{x}^2} \right]$$

where

$$K = \begin{cases} +1 & \text{spherical space} \\ 0 & \text{Euclidean space} \\ -1 & \text{hyperbolical space} \end{cases}$$

Spherical coordinates in space

Recall our spherical coordinates r, θ, ϕ related to the Cartesian ones as

$$\begin{aligned} x &= r \cdot \sin\theta \cdot \cos\phi \\ y &= r \cdot \sin\theta \cdot \sin\phi \\ z &= r \cdot \cos\theta \end{aligned}$$

We saw that

$$\mathrm{d}\vec{x}^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \equiv \mathrm{d}r^2 + r^2\mathrm{d}\Omega.$$

Also, $\vec{x}^2 = r^2$ and $\vec{x} \cdot \mathrm{d}x = r\mathrm{d}r$.

Spherical coordinates

Re-write the metric accordingly:

$$\mathrm{d}s^2 = R^2\left(\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega\right).$$

Evidently, R sets the overall length scale.

This is nice and simple!

Spherical coordinates

Another re-write of the metric: define

$$r = \begin{cases} \sin(\zeta) & \text{for } K = +1 \\ \zeta & \text{for } K = 0 \\ \sinh(\zeta) & \text{for } K = -1 \end{cases}$$

$$ds^2 = R^2 \left(d\zeta^2 + \left\{ \begin{array}{c} \sin^2(\zeta) \\ \zeta^2 \\ \sinh^2(\zeta) \end{array} \right\} d\Omega \right).$$

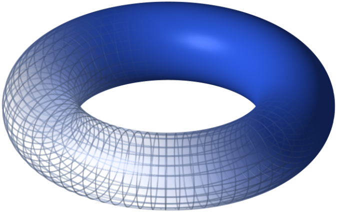
A caveat: global vs. local

The metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\mathrm{d}\Omega.$$

describes space *locally*.

Globally, there is *topology* to consider — e.g. a flat metric can belong to infinite Euclidean space, but also, say, to a torus (a patch of Euclidean space with certain identifications).



⇒ Later on, we will learn of a possibility how a finite universe might be identified (cosmic background radiation)

A caveat: global vs. local

- $K = 0$: 18 topologically different forms of space. Some inifinite, some finite.
- $K = +1$: inifinitely many topologically different forms. All are finite.
- $K = -1$: infinitely many topologically different forms of space. Some inifinite, some finite.

Einstein Equations 1/3

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}$$

General: μ, ν can take on values $0, 1, 2, 3$ for the time direction 0 and space directions $1, 2, 3$.

LHS: $G_{\mu\nu}$ is a combination of second derivatives of the metric coefficients w.r.t. coordinates – embodies a special form of curvature, that is, deviation from flat Minkowski space. $g_{\mu\nu}$ are metric coefficients, Λ is called the cosmological constant.

RHS: Source term. In suitable coordinates, for a homogeneous configuration, the tensor (matrix) $T_{\mu\nu}$ is the energy-momentum tensor (also called stress-energy tensor).

Einstein Equations 2/3

Famous shorthand by John Wheeler: Matter tells space-time how to curve; space-time tells matter how to move.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu}$$

$G_{\mu\nu}$ encodes deviation of free-fall movements from flat space-time: tidal gravitational forces. $T_{\mu\nu}$ encodes information about energy, momentum, pressure, shears etc. associated with the matter. Λ is a constant associated with the space-time in question.

Einstein Equations 3/3

For an ideal fluid (no shear, just pressure) and in suitable (co-moving) coordinates:

$$T_{\mu\nu} = \text{diag}(\rho, p/c^2, p/c^2, p/c^2)$$

with density (includes energy!) ρ and pressure p. Can be used to re-write Einstein's equations as

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}$$

with new contribution to $T_{\mu\nu}$ of

$$\rho_\Lambda = \frac{\Lambda c^4}{8\pi G} = -p_\Lambda/c^2$$

— special form of "energy content": dark energy.

Relativistic model-building

Coupled system of *Einstein's equations* and *equation of state* (specifying the properties of matter):

- General solutions: very messy $\Rightarrow$ *numerical relativity*
- Exact solutions: simple models with symmetry
- Approximation (perturbation theory): e.g. gravitational waves

Each solution of general relativity is automatically a model universe!

What we will need for cosmology

- We must find a metric to describe our cosmological model
- Use gr-freedom of choosing coordinates to choose practical coordinates
- Properties of metric are related to matter content (“energy-momentum tensor”) by Einstein’s equations
- Free-particle movement in that model: geodesics
- Light propagation in that model: null geodesics $\mathrm{d}s^2 = 0$

Exact solutions

Exact solutions are, by necessity *simple model situations*.
Assumption: symmetries!

- Minkowski spacetime (empty)
- Schwarzschild solution (empty w/boundary: black hole)
- Kerr solution (rotating body: rotating bh, gravitomagnetism)
- Friedmann-Lemaître-Robertson-Walker (cosmology, what we'll study now – homogeneous and isotropic)

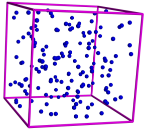
Simple cosmological space-times

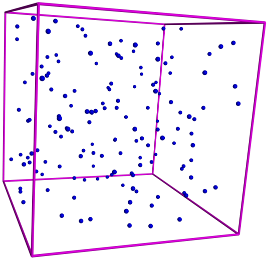
Simplest cosmological models:

- Homogeneous and isotropic universes
- Cosmic substrate: “Galaxy dust”, constant (average) density
- There aren’t that many way a homogeneous universe can change while remaining homogeneous!
- Change that preserves homogeneity/isotropy: $\rho \to \rho(t)$, density can change with time

Changing densities

Changing the density while preserving particle number (simplest model; mass tied to particles): Over time, particles are spread out over an ever larger ($\dot{\rho} < 0$) or ever smaller ($\dot{\rho} > 0$) volume:





Pattern (relative distances) the same — overall scale changes!

Introducing the scale factor

Pattern (relative distances) the same — overall scale changes: All distances between particles change proportional to the same **cosmic scale factor** $a(t)$,

$$d_{ij}(t) = \frac{a(t)}{a(t_R)} \cdot d_{ij}(t_R),$$

for t_R a specific moment in time chosen as reference (in cosmology: usually t_0, the present time).

The spatial metric

We can describe the *pattern* of particles by specifying their positions at any fixed time; distance ratios will remain the same as the scale factor changes.

Choose Cartesian system x, y, z at some reference time t_R. Give each galaxy-particle i the fixed position defined by x, y, z in that system (co-moving coordinates).

Obviously, the unchanging coordinate values cannot reflect the fact that the particles are spreading out (or drawing closer together). Let the metric handle that:

$$(\mathrm{d}s^2)_{space} = a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2).$$

A more general spatial metric

We've seen more general homogeneous metrics ($K = -1, 0, +1$). Robertson (1935, 1936) & Walker (1937) showed these are the *only* possible spatial metrics for a homomgeneous space-time. Generalizing, we choose

$$(\mathrm{d}s^2)_{space} = a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \mathrm{d}\Omega \right),$$

with

Cosmic time and FLRW metric

How to choose time coordinate? Natural for given symmetry: Proper time of each galaxy particle in the cosmic substrate. Simultaneity chosen so that density is indeed constant. Result:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega\right] = -c^2\mathrm{d}\tau^2.$$

with $\mathrm{d}\Omega \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.
This is the **Friedmann-Robertson-Walker-Metric** — unique description for homogeneous and isotropic spaces.
(GR also shows: $r, \theta, \phi =$const. is free motion — the galaxies of the substrate stay where they are.)

Literature on general relativity and cosmology

Lambourne, Robert J.: *Relativity, Gravitation and Cosmology*. Cambridge University Press 2010

d'Inverno, Ray: *Introducing Einstein's Relativity*. Oxford University Press 1992

Rindler, Wolfgang: *Relativity: Special, general and cosmological*. Oxford University Press 2006

Schutz, Bernard: *A first course in general relativity*. Cambridge University Press 1985

Weinberg, Steven: *Gravitation and Cosmology*. Wiley & Sons 1972 [more advanced]