linear and nonlinear structure growth

cosmology lecture (chapter 11)

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1 repetition
2 structure formation equations
3 linearisation
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6 stability
7 summary
• Friedmann-Lemaître cosmologies with matter and dark energy for accelerated expansion
• thermal history of the universe explains element synthesis and the microwave background
• inflation needed for solving the flatness and horizon-problems
• inflationary fluctuations are seed fluctuations for structure formation
• description of Gaussian, homogeneous fluctuations with correlation functions or spectra, assumption of ergodicity
• inflationary perturbations can be seen as fluctuations in the cosmic microwave background
• formation of the cosmic large-scale structure from inflationary perturbations by gravitational instability
structure formation equations

- continuity equation: evolution of the density field due to fluxes
  \[ \frac{\partial}{\partial t} \rho + \text{div}(\rho \vec{v}) = 0 \]  (1)
- Euler equation: evolution of the velocity field due to forces
  \[ \frac{\partial}{\partial t} \vec{v} + \vec{v} \nabla \vec{v} = -\nabla \Phi \]  (2)
- Poisson equation: potential sourced by density field
  \[ \Delta \Phi = 4\pi G \rho \]  (3)

- 3 quantities, 3 equations \(\rightarrow\) solvable
- 2 nonlinearities: \(\rho \vec{v}\) in continuity and \(\vec{v} \nabla \vec{v}\) in Euler-equation

cosmic structure formation

structure formation is a self gravitating, fluid mechanical phenomenon
dynamics with dark matter

dark matter is collisionless (no viscosity and pressure) and interacts gravitationally (non-saturating force)

- dark matter is collisionless $\rightarrow$ no mechanism for microscopic elastic collisions between particles, only interaction by gravity
- derivation of the fluid mechanics equation from the Boltzmann-equation: moments method
  - continuity equation
  - Navier-Stokes equation
  - energy equation
- system of coupled differential equations, and closure relation
- effective description of collisions: viscosity and pressure, but
  - relaxation of objects if there is no viscosity?
  - stabilisation of objects against gravity if there is no pressure?
- Navier-Stokes equation for inviscid fluids is called Euler-equation
collective dynamics: dynamical friction

• dynamical friction emulates viscosity: there is no **microscopic model** for viscosity, but **collective processes** generate an effective viscosity
  • a particle moving through a cloud produces a wake
  • behind the particle, there is a density enhancement
  • density enhancement breaks down particle velocity

• kinetic energy of the incoming object is transformed to unordered random motion
Kelvin-Helmholtz instability

- shear flows become unstable if there are large perpendicular velocity gradients
- generation of vorticity in shear flows by the Kelvin-Helmholtz instability
- absent in the case of dark matter: flow is necessarily laminar
vorticity

• intuitive explanation of the nonlinearity of the Navier-Stokes eqn

\[
\frac{\partial}{\partial t} \boldsymbol{\nu} + \boldsymbol{\nu} \nabla \boldsymbol{\nu} = \nabla p \frac{1}{\rho} - \nabla \Phi + \mu \Delta \boldsymbol{\nu}
\]  \hspace{1cm} (4)

• vorticity equation: \( \boldsymbol{\omega} \equiv \text{rot} \boldsymbol{\nu} \)

\[
\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nu} \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \nabla \boldsymbol{\nu} - \boldsymbol{\omega} \text{div} \boldsymbol{\nu} + \frac{1}{\rho^2} \nabla p \times \nabla \rho + \mu \Delta \boldsymbol{\omega}
\]  \hspace{1cm} (5)

material derivative \hspace{1cm} tilting \hspace{1cm} compression \hspace{1cm} baroclinic \hspace{1cm} diffusion

• generation of vorticity by
  • pressure gradients non-parallel to density gradients
  • viscous stresses

→ not present in the case of collisionless dark matter
→ gravity as a conservative force is not able to induce vorticity

• vorticity equation is a nonlinear diffusion equation, vorticity is advected by its own induced velocity field
regimes of structure formation

**look at overdensity field** $\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$, with $\bar{\rho} = \Omega_m \rho_{\text{crit}}$

- analytical calculations are possible in the regime of linear structure formation, $\delta \ll 1$
  $\rightarrow$ homogeneous growth, dependence on dark energy, number density of objects

- transition to non-linear structure growth can be treated in perturbation theory (difficult!), $\delta \sim 1$
  $\rightarrow$ first numerical approaches (Zel’dovich approximation), directly solvable for geometrically simple cases (spherical collapse)

- non-linear structure formation at late times, $\delta > 1$
  $\rightarrow$ higher order perturbation theory (even more difficult), ultimately: direct simulation with $n$-body codes
linearisation: perturbation theory for $\delta \ll 1$

- move from physical to comoving frame, related by scale-factor $a$
- use density $\delta = \Delta \rho / \rho$ and comoving velocity $\vec{u} = \vec{\nu} / a$
  - **linearised continuity equation**:
    $$ \frac{\partial}{\partial t} \delta + \text{div} \vec{u} = 0 $$
  - **linearised Euler equation**: evolve momentum
    $$ \frac{\partial}{\partial t} \vec{u} + 2H(a)\vec{u} = -\frac{\nabla \Phi}{a^2} $$
  - **Poisson equation**: generate potential
    $$ \Delta \Phi = 4\pi G \rho_0 a^2 \delta $$

**question**

derive the linearised equations by substituting a perturbative series $\rho = \rho_0 (1 + \delta)$ for all quantities, in the comoving frame
growth equation

- structure formation is homogeneous in the linear regime, all spatial derivatives drop out
- combine continuity, Jeans- and Poisson-eqn. for differential equation for the temporal evolution of $\delta$:
  \[
  \frac{d^2\delta}{da^2} + \frac{1}{a} \left( 3 + \frac{d \ln H}{d \ln a} \right) \frac{d\delta}{da} = \frac{3\Omega_M(a)}{2a^2} \delta
  \]

  (6)

- growth function $D_+(a) \equiv \delta(a)/\delta(a = 1)$ (growing mode)
  - position and time dependence separated: $\delta(\vec{x}, a) = D_+(a)\delta_0(\vec{x})$
  - in Fourier-space modes grows independently: $\delta(\vec{k}, a) = D_+(a)\delta_0(\vec{k})$

- for standard gravity, the growth function is determined by $H(a)$

**question**

derive the growth function $D_+$ with $t$ and with $a$ as time variables
• two terms in growth equation:
  - source $Q(a) = \Omega_m(a)$: large $\Omega_m(a)$ make the grav. fields strong
  - dissipation $S(a) = 3 + d \ln H / d \ln a$: structures grow if their dynamical time scale is smaller than the Hubble time scale $1/H(a)$
growth function

\[ D_+(a) \text{ for } \Omega_m = 1 \text{ (dash-dotted), for } \Omega_\Lambda = 0.7 \text{ (solid) and } \Omega_k = 0.7 \text{ (dashed)} \]

- density field grows \( \propto a \) in \( \Omega_m = 1 \) universes, faster if \( w < 0 \)

**question**

show that \( D_+(a) = a \) is a solution for \( \Omega_m = 1 \). what would be the solution in the radiation dominated epoch?
nonlinear density fields

\[ \Lambda \text{CDM} \quad \text{SCDM (} \Omega_m = 1 \text{)} \]

source: Virgo consortium

- dark energy influences nonlinear structure formation
- how does nonlinear structure formation change the statistics of the density field?
mode coupling

- linear regime structure formation: homogeneous growth
  \[ \delta(\vec{x}, a) = D_+(a)\delta_0(\vec{x}) \rightarrow \delta(\vec{k}, a) = D_+(a)\delta_0(\vec{k}) \]  
  \[(7)\]

- separation fails if the growth is nonlinear, because a void can’t get more empty than \( \delta = -1 \), but a cluster can grow to \( \delta \approx 200 \)
  \[ \delta(\vec{x}, a) = D_+(a, \vec{x})\delta_0(\vec{x}) \]  
  \[(8)\]

- product of two \( \vec{x} \)-dependent quantities in real space \( \rightarrow \) convolution in Fourier space:
  \[ \delta(\vec{k}, a) = \int d^3k' D_+(a, \vec{k} - \vec{k}')\delta_0(\vec{k}') \]  
  \[(9)\]

- \( k \)-modes do not evolve independently: **mode coupling**

how that products of functions in real space become convolutions in Fourier-space
perturbation theory

• perturbative series in density field:

\[ \delta(\mathbf{x}, a) = D_+(a)\delta^{(1)}(\mathbf{x}) + D_+^2(a)\delta^{(2)}(\mathbf{x}) + D_+^3(a)\delta^{(3)}(\mathbf{x}) + \ldots \]  

(10)

• lowest order:

\[ \delta^{(2)}(\mathbf{k}) = \int \frac{d^3p}{(2\pi)^3} M_2(\mathbf{k} - \mathbf{p}, \mathbf{p})\delta(\mathbf{p})\delta(|\mathbf{k} - \mathbf{p}|) \]  

(11)

• with mode coupling

\[ M_2(\mathbf{p}, \mathbf{q}) = \frac{10}{7} + \frac{\mathbf{p} \cdot \mathbf{q}}{pq} \left( \frac{p}{q} + \frac{q}{p} \right) + \frac{4}{7} \left( \frac{\mathbf{p} \cdot \mathbf{q}}{pq} \right)^2 \]  

(12)

• properties:
  - time-independent, no scale \( \mathbf{p}_0 \)
  - strongest coupling if \( \mathbf{p} = \mathbf{q} \)
  - some coupling of modes \( \mathbf{p} \perp \mathbf{q} \)
  - no coupling if \( \mathbf{p} = -\mathbf{q} \)
homogeneity, linearity and Gaussianity

...almost the same thing in structure formation!

• linearity
  • eqns can be linearised: $|\delta| \ll 1$
  • linearisation fails: $|\delta| \approx 1$

• homogeneity
  • homogeneous: $\delta(\vec{x}, a) = D_+(a)\delta(\vec{x}, a = 1)$
  • inhomogeneous: $\delta(\vec{x}, a) = D_+(\vec{x}, a)\delta(\vec{x}, a = 1)$

• Gaussianity (with central limit theorem)
  • Gaussian amplitude distribution $p(\delta)d\delta$
  • non-Gaussian (lognormal) distribution $p(\delta)d\delta$

mode coupling

easiest way to visualise: resonance phenomenon
nonlinearity triangle

- linearity, homogeneity and Gaussianity imply each other
- nonlinear structure formation breaks homogeneity and produces non-Gaussian statistics
- mode coupling - can be described in perturbation theory

linear and nonlinear structure growth

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link between dynamics and statistics

- nonlinear structure formation couples modes
- superposition of various $k$-modes (not independent anymore) generate a non-Gaussian density field
- non-Gaussian density field:
  - odd moments are not necessarily zero
  - even moments are not powers of the variance
- finite correlation length: $n$-point correlation functions
  - 3-point-function: bispectrum
  - 4-point-function: trispectrum

higher order correlations quickly become unpractical, and are really difficult to determine
nonlinear CDM spectrum $P(k)$

- fit to numerical data, $z = 9, 4, 1, 0$, normalised on large scales
- extra power on large scales, time dependent, saturates
- on top of scaling $P(k, a) \propto D^2_+(a)$
quantification of non-Gaussianities: bispectrum

- bispectrum (3-point function) quantifies nonlinearity to lowest order
- configuration dependence: compare arbitrary triangle to equilateral triangle, keeping base fixed:

\[
R_{\ell_3}(\ell_1, \ell_2) = \frac{\ell_1 \ell_2}{\ell_3^2} \sqrt{\frac{B(\ell_1, \ell_2, \ell_3)}{B(\ell_3, \ell_3, \ell_3)}}
\]  (13)
$n$-body simulations of structure formation

- basic issue: gravity is long-ranged, for each particle the gravitational force of all other particle needs to be summed up, complexity $n^2$
- algorithmic challenge to break down $n^2$-scaling
  - particle-mesh
  - particle$^3$-mesh
  - tree-codes
  - tree-particle mesh
Zel’dovich-approximation

- evolution of perturbation in the translinear regime
- idea: follow trajectories of particles that accumulate in a region and produce a density fluctuation
- physical position \( \vec{r} \) (Euler) can be related to initial position \( \vec{q} \) (Lagrange)

\[
\vec{x} = \frac{\vec{r}(t)}{a} = \vec{q} + D_+(t) \nabla \Psi(\vec{q})
\]  \hspace{1cm} (14)

- two contributions: Hubble-flow and local deviation, expressed by displacement field \( \Psi(\vec{q}) \)
- displacement field \( \Psi \) is a solution to Poisson eqn. \( \Delta \Psi = \delta \)
- evolution dominated by overall potential, not by self-gravity

question

can \( \delta \) become infinite in the Zel’dovich-approximation? what happens in Nature?
Zel’ dovich-approximation: quick realisation

time sequence of structure formation in a dark energy cosmology

- formation of sheets and filaments
- very fast computational scheme (above pic: seconds!!)
- can’t use Zel’dovich approximation, if trajectories cross
- no relaxation (collapsing sphere would reexpand to original radius)
Zel’dovich: comparison to exact solution

- reexpanding structures, no dissipation, no formation of objects
- qualitative agreement on large scales, small densities
Angular momentum of galaxies

- Vorticity can’t be generated in inviscid fluids
- Flow is laminar
- Initial vorticity decreases $\propto \frac{1}{a}$
angular momentum: tidal shearing

- non-constant displacement mapping across protogalactic cloud
- tidal forces $\partial_i \partial_j \Psi$ set protogalactic cloud into rotation
- in addition: anisotropic deformation (not drawn!)
- gravitational collapse: non-simply connected fields
tidal shearing in Zel’dovich-approximation

- current paradigm: galactic haloes acquire angular momentum by tidal shearing (White 1984)

\[ \mathbf{\dot{L}} \simeq \rho_0 a^5 \int_{V_L} d^3 \mathbf{q} (\mathbf{q} - \bar{\mathbf{q}}) \times \dot{\mathbf{x}} \]  

(15)

- tidal shearing can be described in Zel’dovich approximation

\[ \mathbf{x}(\mathbf{q}, t) = \mathbf{q} - D_+(t) \nabla \Psi(\mathbf{q}) \rightarrow \dot{\mathbf{x}} = -\dot{D}_+ \nabla \Psi \]  

(16)

- 2 relevant quantities: inertia \( I_{\alpha\beta} \) and shear \( \Psi_{\alpha\beta} \)

\[ L_\alpha = a^2 \dot{D}_+ \epsilon_{\alpha\beta\gamma} I_{\beta\sigma} \Psi_{\sigma\gamma} \]  

(17)

- tidal shear \( \Psi_{\alpha\beta} = \partial_\alpha \partial_\beta \Psi \), derived from Zel’dovich displacement field \( \Psi \propto \Phi \), solution to \( \Delta \Psi = \delta \)
tidal interaction with the large-scale structure

- dynamics described by Zel’dovich approximation (lowest order)
- \( L_\alpha = a^2 \dot{D}_+ \epsilon_{\alpha\beta\gamma} I_{\beta\sigma} \Psi_{\sigma\gamma} \), with inertia \( I \) and gravitational shear \( \Psi \)
- define \( X = I\Psi \), split up \( X = X^+ + X^- \):
  - \( L \propto X^- = \frac{1}{2} [I, \Psi] \), misalignment between shear and inertia, skewed eigensystems necessary for inducing rotation
  - \( X^+ = \frac{1}{2} \{I, \Psi\} \) causes an anisotropic deformation
gravothermal instability: thermal energy

- consider gravitationally bound system, exchanging (thermal) energy with environment
  1. energy is removed from a self-gravitating object, on a time-scale
    \[ t_{\text{remove}} \gg t_{\text{dyn}} \]
  2. system assumes a new equilibrium state deeper inside its own potential well (quasi-stationary, no relaxation)
  3. release of gravitational binding energy, particles speed up
  4. velocity dispersion (temperature) rises

- removal of thermal energy \(\rightarrow\) increase in temperature
- gravitationally bound systems have a negative specific heat

question

in what way can you get a self-gravitating system to cool down?

question

could one use such systems as an unlimited source of energy?
negative specific heat: virial theorem

- look at the **kinetic energy** \( T = \sum_i^n m/2\nu_i^2 \) for a system of \( n \) particles

\[
\frac{\partial T}{\partial \nu_i} = m\nu_i \quad \rightarrow \quad \sum_i^n \frac{\partial T}{\partial \nu_i} \nu_i = 2T
\]  

(18)

- if we introduce **momenta** \( p_i = \frac{\partial T}{\partial \nu_i} \):

\[
2T = \sum_i^n p_i\nu_i = \frac{d}{dt} \sum_i^n p_ir_i - \sum_i r_i\dot{p}_i
\]  

(19)

with particle positions \( r_i \) with \( \dot{r}_i = \nu_i \)

- perform **time averaging**

\[
\langle \psi \rangle \equiv \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_0^{\Delta t} dt \psi(t)
\]  

(20)
negative specific heat: virial theorem

- if $\psi(t)$ is the derivative of a **bounded function** $\Psi$, this average vanishes:

$$
\langle \psi \rangle = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_0^{\Delta t} dt \frac{d\Phi}{dt} = \lim_{\Delta t \to \infty} \frac{\Psi(\Delta t) - \Psi(0)}{\Delta t} = 0 \quad (21)
$$

- the **virial** $\sum_i r_i p_i$ is bounded, so its average of its derivative vanishes

- if the system is **Newtonian**, $\dot{p}_i = -\partial \Phi / \partial r_i$

$$
2\langle T \rangle = \left\langle \sum_{i}^{n} r_i \frac{\partial \Phi}{\partial r_i} \right\rangle \quad (22)
$$

- if the potential is a **homogeneous** function of order $k$, $\Phi(\alpha r) = \alpha^k \Phi(r)$, one gets:

$$
2\langle T \rangle = k\langle \Phi \rangle \quad (23)
$$
negative specific heat: virial theorem

- substituting the total energy $E$ gives $\langle T \rangle + \langle \Phi \rangle = E$ and therefore
  $$\langle T \rangle = \frac{2}{k + 2} E \quad \text{and} \quad \langle \Phi \rangle = \frac{k}{k + 2} E$$  
  \hspace{1cm} (24)

- for the Newtonian gravitational potential $\Phi \propto 1/r$ the homogeneity parameter is $k = -1$: $2\langle T \rangle = -\langle \Phi \rangle$, or equivalently
  $$\langle T \rangle = 2E \quad \text{and} \quad \langle \Phi \rangle = -E$$  
  \hspace{1cm} (25)

- if one removes energy, the system would be more tightly bound and $E$ would be more negative

- as a consequence, the particles would need to speed up and the temperature increases

question

imagine particles in a system would be bound by a harmonic potential $\Phi \propto r^2$. would this system have positive or negative specific heat?
gravothermal instability: particles

globular cluster Omega Centauri, source: Loke Kun Tan

- kinetic energy of a star fluctuates, can get gravitationally unbound
- star leaves cluster on parabolic orbit, does not take away energy
- gravitational binding energy distributed among fewer stars
- system heats up by evaporating stars, eventually disintegrates

question
when does this process stop? what’s the final state?
gravothermal instability: particles

• for a gravitationally bound system, we would write $E = -\langle \Phi \rangle$ with the potential energy $\Phi = GM^2/R$

• in the evaporation process, the total energy is approximately conserved, so

$$\frac{dE}{dt} = 0 = \frac{2GM}{R} \frac{dM}{dt} - \frac{GM^2}{R^2} \frac{dR}{dt} \quad \Rightarrow \quad \frac{2R}{M} \frac{dM}{dt} = \frac{dR}{dt}$$

(26)

• let’s assume a simple law for the mass loss:

$$\frac{dM}{dt} = -\frac{M}{\tau}$$

(27)

which leads to a decaying exponential $M(t) = M_0 \exp(-t/\tau)$

question

can you combine these equations for a differential equation for $R(t)$ and solve it? what makes it consistent with the $M(t)$-solution?
The large-scale distribution of matter in the universe forms by gravitational instability. It is described by the continuity equation, Euler-equation (with dark matter being collisionless), and the Poisson equation (Newtonian gravity). Linearisation is valid when $\delta \ll 1$, leading to a growth equation where:

- growth is homogeneous
- it conserves all statistical properties of the field, especially Gaussianity

In the nonlinear regime $\delta \gg 1$, the linearisation fails. Growth becomes inhomogeneous, Gaussianity is violated by mode coupling, galaxy rotation is explained by tidal interaction, and haloes form by gravitational collapse, but their stability is difficult to understand.