linear and nonlinear structure growth

cosmology lecture (chapter 11)

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outline

repetition

- Friedmann-Lemaître cosmologies with matter and dark energy for accelerated expansion
- thermal history of the universe explains element synthesis and the microwave background
- inflation needed for solving the flatness and horizon-problems
- inflationary fluctuations are seed fluctuations for structure formation
- description of Gaussian, homogeneous fluctuations with correlation functions or spectra, assumption of ergodicity
- inflationary perturbations can be seen as fluctuations in the cosmic microwave background
- formation of the cosmic large-scale structure from inflationary perturbations by gravitational instability

structure formation equations

cosmic structure formation

structure formation is a self gravitating, fluid mechanical phenomenon

- continuity equation: evolution of the density field due to fluxes

$$\frac{\partial}{\partial t}\rho + \mathrm{div}(\rho\vec{\upsilon}) = 0 \quad (1)$$

- Euler equation: evolution of the velocity field due to forces

$$\frac{\partial}{\partial t}\vec{\upsilon} + \vec{\upsilon}\nabla\vec{\upsilon} = -\nabla\Phi \quad (2)$$

- Poisson equation: potential sourced by density field

$$\Delta\Phi = 4\pi G\rho \quad (3)$$

- 3 quantities, 3 equations → solvable
- 2 nonlinearities: $\rho\vec{\upsilon}$ in continuity and $\vec{\upsilon}\nabla\vec{\upsilon}$ in Euler-equation

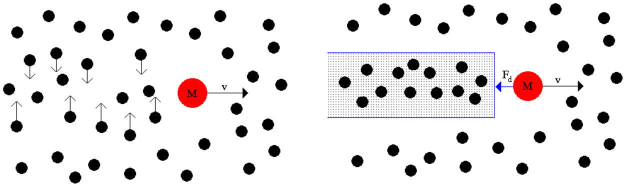
viscosity and pressure

dynamics with dark matter

dark matter is collisionless (no viscosity and pressure) and interacts gravitationally (non-saturating force)

- dark matter is collisionless → no mechanism for microscopic elastic collisions between particles, only interaction by gravity
- derivation of the fluid mechancis equation from the Boltzmann-equation: **moments method**
  - continuity equation
  - Navier-Stokes equation
  - energy equation
- system of coupled differential equations, and closure relation
- effective description of collisions: viscosity and pressure, but
  - relaxation of objects if there is no viscosity?
  - stabilisation of objects against gravity if there is no pressure?
- Navier-Stokes equation for inviscid fluids is called **Euler-equation**

collective dynamics: dynamical friction



source: J. Schombert

- dynamical friction emulates viscosity: there is no **microscopic model** for viscosity, but **collective processes** generate an effective viscosity
  - a particle moving through a cloud produces a wake
  - behind the particle, there is a density enhancement
  - density enhancement breaks down particle velocity
- kinetic energy of the incoming object is transformed to unordered random motion

Kelvin-Helmholtz instability

- shear flows become unstable if there are large perpendicular velocity gradients
- generation of vorticity in shear flows by the **Kelvin-Helmholtz instability**
- absent in the case of dark matter: flow is necessarily laminar

vorticity

- intuitive explanation of the nonlinearity of the Navier-Stokes eqn

$$\frac{\partial}{\partial t}\vec{\upsilon} + \vec{\upsilon}\nabla\vec{\upsilon} = \frac{\nabla p}{\rho} - \nabla\Phi + \mu\Delta\vec{\upsilon} \tag{4}$$

- vorticity equation: $\vec{\omega} \equiv \mathrm{rot}\vec{\upsilon}$

$$\underbrace{\frac{\partial\vec{\omega}}{\partial t} + \vec{\upsilon}\nabla\vec{\omega}}_{\text{material derivative}} = \underbrace{\vec{\omega}\nabla\vec{\upsilon}}_{\text{tilting}} - \underbrace{\vec{\omega}\mathrm{div}\vec{\upsilon}}_{\text{compression}} + \underbrace{\frac{1}{\rho^2}\nabla p \times \nabla\rho}_{\text{baroclinic}} + \underbrace{\mu\Delta\vec{\omega}}_{\text{diffusion}} \tag{5}$$

- generation of vorticity by
  - pressure gradients non-parallel to density gradients
  - viscous stresses

$\rightarrow$ **not present in the case of collisionless dark matter**
$\rightarrow$ **gravity as a conservative force is not able to induce vorticity**

- vorticity equation is a nonlinear diffusion equation, vorticity is advected by its own induced velocity field

regimes of structure formation

look at overdensity field $\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$, with $\bar{\rho} = \Omega_m \rho_{\mathrm{crit}}$

- analytical calculations are possible in the regime of linear structure formation, $\delta \ll 1$
  $\rightarrow$ homogeneous growth, dependence on dark energy, number density of objects
- transition to non-linear structure growth can be treated in perturbation theory (difficult!), $\delta \sim 1$
  $\rightarrow$ first numerical approaches (Zel'dovich approximation), directly solvable for geometrically simple cases (spherical collapse)
- non-linear structure formation at late times, $\delta > 1$
  $\rightarrow$ higher order perturbation theory (even more difficult), ultimately: direct simulation with n-body codes

linearisation: perturbation theory for $\delta \ll 1$

- move from physical to comoving frame, related by scale-factor a
- use density $\delta = \Delta\rho/\rho$ and comoving velocity $\vec{u} = \vec{\upsilon}/a$
  - **linearised continuity equation**:

$$\frac{\partial}{\partial t}\delta + \mathrm{div}\vec{u} = 0$$

  - **linearised Euler equation**: evolve momentum

$$\frac{\partial}{\partial t}\vec{u} + 2H(a)\vec{u} = -\frac{\nabla\Phi}{a^2}$$

  - **Poisson equation**: generate potential

$$\Delta\Phi = 4\pi G\rho_0 a^2\delta$$

question

derive the linearised equations by subsituting a perturbative series $\rho = \rho_0(1+\delta)$ for all quantities, in the comoving frame

growth equation

- structure formation is homogeneous in the linear regime, all spatial derivatives drop out
- combine continuity, Jeans- and Poisson-eqn. for differential equation for the temporal evolution of δ:

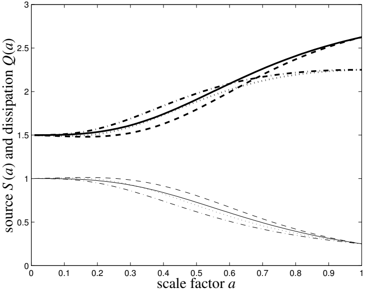
$$\frac{\mathrm{d}^2\delta}{\mathrm{d}a^2} + \frac{1}{a}\left(3 + \frac{\mathrm{d}\ln H}{\mathrm{d}\ln a}\right)\frac{\mathrm{d}\delta}{\mathrm{d}a} = \frac{3\Omega_M(a)}{2a^2}\delta \tag{6}$$

- growth function $D_+(a) \equiv \delta(a)/\delta(a=1)$ (growing mode)
  - position and time dependence separated: $\delta(\vec{x}, a) = D_+(a)\delta_0(\vec{x})$
  - in Fourier-space modes grows independently: $\delta(\vec{k}, a) = D_+(a)\delta_0(\vec{k})$
- for standard gravity, the growth function is determined by $H(a)$

question

derive the growth function D_+ with t and with a as time variables

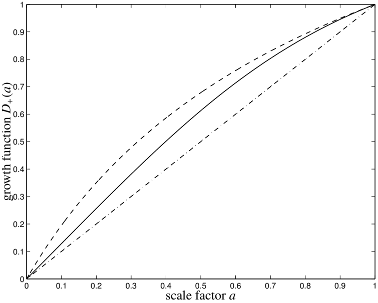
terms in the growth equation



source (thin line) and dissipation (thick line)

- two terms in growth equation:
  - source $Q(a) = \Omega_m(a)$: large $\Omega_m(a)$ make the grav. fields strong
  - dissipation $S(a) = 3 + \mathrm{d}\ln H/\mathrm{d}\ln a$: structures grow if their dynamical time scale is smaller than the Hubble time scale $1/H(a)$

growth function



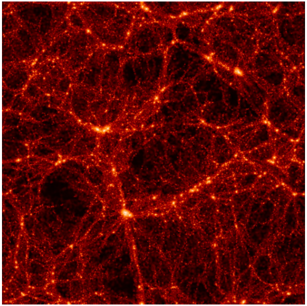
$D_+(a)$ **for** $\Omega_m = 1$ **(dash-dotted), for** $\Omega_\Lambda = 0.7$ **(solid) and** $\Omega_k = 0.7$ **(dashed)**

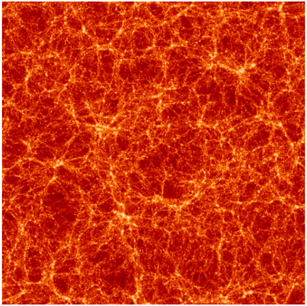
- density field grows $\propto a$ in $\Omega_m = 1$ universes, faster if $w < 0$

question

show that $D_+(a) = a$ is a solution for $\Omega_m = 1$. what would be the solution in the radiation dominated epoch?

nonlinear density fields





ΛCDM SCDM ($\Omega_m = 1$)

source: Virgo consortium

- dark energy influences nonlinear structure formation
- how does nonlinear structure formation change the statistics of the density field?

mode coupling

- linear regime structure formation: homogeneous growth

$$\delta(\vec{x}, a) = D_+(a)\delta_0(\vec{x}) \rightarrow \delta(\vec{k}, a) = D_+(a)\delta_0(\vec{k}) \tag{7}$$

- separation fails if the growth is nonlinear, because a void can't get more empty than $\delta = -1$, but a cluster can grow to $\delta \simeq 200$

$$\delta(\vec{x}, a) = D_+(a, \vec{x})\delta_0(\vec{x}) \tag{8}$$

- product of two $\vec{x}$-dependent quantities in real space $\rightarrow$ convolution in Fourier space:

$$\delta(\vec{k}, a) = \int \mathrm{d}^3k' D_+(a, \vec{k} - \vec{k}')\delta_0(\vec{k}') \tag{9}$$

- k-modes do not evolve independently: **mode coupling**

s

how that products of functions in real space become convolutions in Fourier-space

perturbation theory

- perturbative series in density field:

$$\delta(\vec{x},a) = D_+(a)\delta^{(1)}(\vec{x}) + D_+^2(a)\delta^{(2)}(\vec{x}) + D_+^3(a)\delta^{(3)}(\vec{x}) + \ldots \tag{10}$$

- lowest order:

$$\delta^{(2)}(\vec{k}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, M_2(\vec{k}-\vec{p},\vec{p})\delta(\vec{p})\delta(|\vec{k}-\vec{p}|) \tag{11}$$

- with mode coupling

$$M_2(\vec{p},\vec{q}) = \frac{10}{7} + \frac{\vec{p}\vec{q}}{pq}\left(\frac{p}{q}+\frac{q}{p}\right) + \frac{4}{7}\left(\frac{\vec{p}\vec{q}}{pq}\right)^2 \tag{12}$$

- properties:
  - time-independent, no scale $\vec{p}_0$
  - strongest coupling if $\vec{p} = \vec{q}$
  - some coupling of modes $\vec{p} \perp \vec{q}$
  - no coupling if $\vec{p} = -\vec{q}$

homogeneity, linearity and Gaussianity

homogeneity, linearity and Gaussianity

...almost the same thing in structure formation!

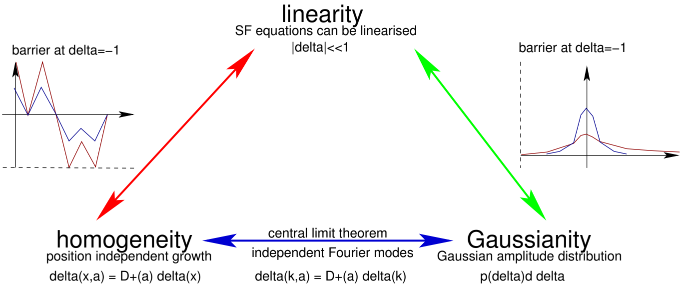
- linearity
  - eqns can be linearised: $|\delta| \ll 1$
  - linearisation fails: $|\delta| \simeq 1$
- homogeneity
  - homogeneous: $\delta(\vec{x}, a) = D_+(a)\delta(\vec{x}, a = 1)$
  - inhomogeneous: $\delta(\vec{x}, a) = D_+(\vec{x}, a)\delta(\vec{x}, a = 1)$
- Gaussianity (with central limit theorem)
  - Gaussian amplitude distribution $p(\delta)\mathrm{d}\delta$
  - non-Gaussian (lognormal) distribution $p(\delta)\mathrm{d}\delta$

mode coupling

easiest way to visualise: resonance phenomenon

nonlinearity triangle

- linearity, homogeneity and Gaussianity imply each other
- nonlinear structure formation breaks homogeneity and produces non-Gaussian statistics
- mode coupling - can be described in perturbation theory

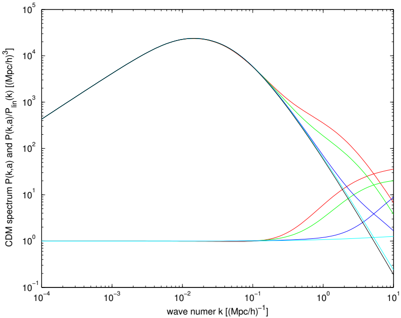


link between dynamics and statistics

- nonlinear structure formation couples modes
- superposition of various k-modes (not independent anymore) generate a non-Gaussian density field
- non-Gaussian density field:
  - odd moments are not necessarily zero
  - even moments are not powers of the variance
- finite correlation length: n-point correlation functions
  - 3-point-function: bispectrum
  - 4-point-function: trispectrum

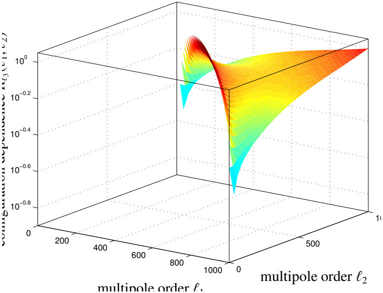
higher order correlations quickly become unpractical, and are really difficult to determine

nonlinear CDM spectrum $P(k)$



- fit to numerical data, $z = 9, 4, 1, 0$, normalised on large scales
- extra power on large scales, time dependent, saturates
- on top of scaling $P(k, a) \propto D_+^2(a)$

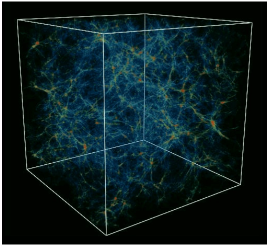
quantification of non-Gaussianities: bispectrum



- bispectrum (3-point function) quantifies nonlinearity to lowest order
- configuration dependence: compare arbitrary triangle to equilateral triangle, keeping base fixed:

$$R_{\ell_3}(\ell_1, \ell_2) = \frac{\ell_1 \ell_2}{\ell_3^2} \sqrt{\left|\frac{B(\ell_1, \ell_2, \ell_3)}{B(\ell_3, \ell_3, \ell_3)}\right|} \tag{13}$$

n-body simulations of structure formation



- basic issue: gravity is long-ranged, for each particle the gravitational force of all other particle needs to be summed up, **complexity n^2**
- algorithmic challenge to break down n^2-scaling
  - **particle-mesh**
  - **particle^3-mesh**
  - **tree**-codes
  - **tree-particle mesh**

Zel’dovich-approximation

- evolution of perturbation in the translinear regime
- idea: follow trajectories of particles that accumulate in a region and produce a density fluctuation
- physical position $\vec{r}$ (Euler) can be related to initial position $\vec{q}$ (Lagrange)

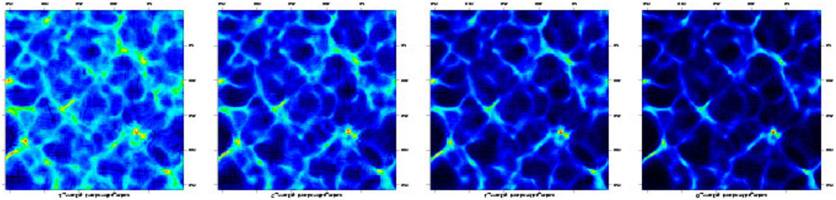
$$\vec{x} = \frac{\vec{r}(t)}{a} = \vec{q} + D_+(t)\nabla\Psi(\vec{q}) \qquad (14)$$

- two contributions: Hubble-flow and local deviation, expressed by displacement field $\Psi(\vec{q})$
- displacement field Ψ is a solution to Poisson eqn. $\Delta\Psi = \delta$
- evolution dominated by overall potential, not by self-gravity

question

can δ become infinite in the Zel’dovich-approximation? what happens in Nature?

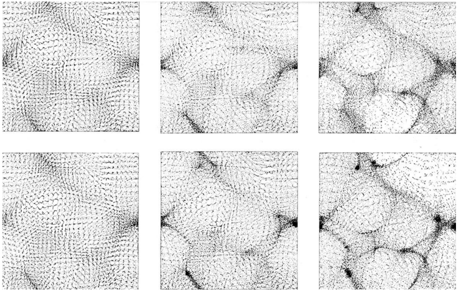
Zel'dovich-approximation: quick realisation



time sequence of structure formation in a dark energy cosmology

- formation of sheets and filaments
- very fast computational scheme (above pic: seconds!!)
- can't use Zel'dovich approximation, if trajectories cross
- no relaxation (collapsing sphere would reexpand to orginial radius)

Zel’dovich: comparision to exact solution



comparison between Zeldovich and exact solution, source: N. Wright

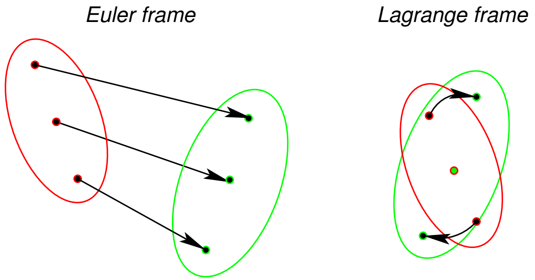
- reexpanding structures, no dissipation, no formation of objects
- qualitative agreement on large scales, small densities

angular momentum of galaxies

galaxy M81, HST image

- vorticity can't be generated in inviscid fluids
- flow is laminar
- initial vorticity decreases $\propto 1/a$

angular momentum: tidal shearing



- non-constant displacement mapping across protogalactic cloud
- tidal forces $\partial_i \partial_j \Psi$ set protogalactic cloud into rotation
- in addition: anisotropic deformation (not drawn!)
- gravitational collapse: non-simply connected fields

tidal shearing in Zel'dovich-approximation

- current paradigm: galactic haloes acquire angular momentum by tidal shearing (White 1984)

$$\vec{L} \simeq \rho_0 a^5 \int_{V_L} \mathrm{d}^3 q (\vec{q} - \bar{q}) \times \dot{\vec{x}} \tag{15}$$

- tidal shearing can be described in Zel'dovich approximation

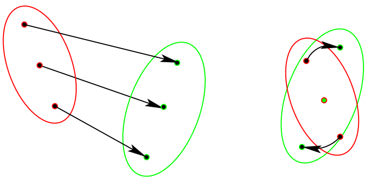
$$\vec{x}(\vec{q}, t) = \vec{q} - D_+(t) \nabla\Psi(\vec{q}) \rightarrow \dot{\vec{x}} = -\dot{D}_+ \nabla\Psi \tag{16}$$

- 2 relevant quantities: inertia $I_{\alpha\beta}$ and shear $\Psi_{\alpha\beta}$

$$L_\alpha = a^2 \dot{D}_+ \epsilon_{\alpha\beta\gamma} I_{\beta\sigma} \Psi_{\sigma\gamma} \tag{17}$$

- tidal shear $\Psi_{\alpha\beta} = \partial_\alpha \partial_\beta \Psi$, derived from Zel'dovich displacement field $\Psi \propto \Phi$, solution to $\Delta\Psi = \delta$

tidal interaction with the large-scale structure



- dynamics described by Zel'dovich approximation (lowest order)
- $L_\alpha = a^2 \dot{D}_+ \epsilon_{\alpha\beta\gamma} I_{\beta\sigma} \Psi_{\sigma\gamma}$, with inertia I and gravitational shear Ψ
- define $\mathbf{X} = \mathbf{I\Psi}$, split up $\mathbf{X} = \mathbf{X}^+ + \mathbf{X}^-$:
  - $L \propto \mathbf{X}^- = \frac{1}{2}[\mathbf{I}, \mathbf{\Psi}]$, **misalignment between shear and inertia**, skewed eigensystems necessary for inducing rotation
  - $\mathbf{X}^+ = \frac{1}{2}\{\mathbf{I}, \mathbf{\Psi}\}$ causes an **anisotropic deformation**

gravothermal instability: thermal energy

- consider gravitationally bound system, exchanging (thermal) energy with environment
    1. energy is removed from a self-gravitating object, on a time-scale $t_\mathrm{remove} \gg$ dynamical time-scale t_dyn
    2. system assumes a new equilibrium state *deeper* inside its own potential well (quasi-stationary, no relaxation)
    3. release of gravitational binding energy, particles speed up
    4. velocity dispersion (temperature) rises
- removal of thermal energy $\rightarrow$ increase in temperature
- gravitationally bound systems have a **negative specific heat**

question

in what way can you get a self-gravitating system to cool down?

question

could one use such systems as an unlimited source of energy?

negative specific heat: virial theorem

- look at the **kinetic energy** $T = \sum_i^n m/2\upsilon_i^2$ for a system of n particles

$$\frac{\partial T}{\partial \upsilon_i} = m\upsilon_i \quad \rightarrow \quad \sum_i^n \frac{\partial T}{\partial \upsilon_i}\upsilon_i = 2T \tag{18}$$

- if we introduce **momenta** $p_i = \partial T / \partial \upsilon_i$:

$$2T = \sum_i^n p_i \upsilon_i = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i^n p_i r_i - \sum_i r_i \dot{p}_i \tag{19}$$

with particle positions r_i with $\dot{r}_i = \upsilon_i$

- perform **time averaging**

$$\langle \psi \rangle \equiv \lim_{\Delta t \to \infty} \frac{1}{\Delta t}\int_0^{\Delta t} \mathrm{d}t\, \psi(t) \tag{20}$$

negative specific heat: virial theorem

- if $\psi(t)$ is the derivative of a **bounded function** Ψ, this average vanishes:

$$\langle\psi\rangle = \lim_{\Delta t\to\infty} \frac{1}{\Delta t}\int_0^{\Delta t} \mathrm{d}t\, \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \lim_{\Delta t\to\infty} \frac{\Psi(\Delta t)-\Psi(0)}{\Delta t} = 0 \qquad (21)$$

- the **virial** $\sum_i r_i p_i$ is bounded, so its average of its derivative vanishes
- if the system is **Newtonian**, $\dot{p}_i = -\partial\Phi/\partial r_i$

$$2\langle T\rangle = \left\langle \sum_i^n r_i \frac{\partial\Phi}{\partial r_i}\right\rangle \qquad (22)$$

- if the potential is a **homogeneous** function of order k, $\Phi(\alpha r) = \alpha^k\Phi(r)$, one gets:

$$2\langle T\rangle = k\langle\Phi\rangle \qquad (23)$$

negative specific heat: virial theorem

- substituting the total energy E gives $\langle T\rangle + \langle\Phi\rangle = E$ and therefore

$$\langle T\rangle = \frac{2}{k+2}E \quad \text{and} \quad \langle\Phi\rangle = \frac{k}{k+2}E \tag{24}$$

- for the Newtonian gravitational potential $\Phi \propto 1/r$ the homogeneity parameter is $k = -1$: $2\langle T\rangle = -\langle\Phi\rangle$, or equivalently

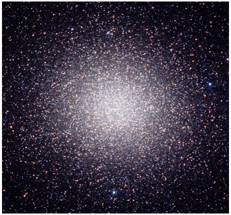
$$\langle T\rangle = 2E \quad \text{and} \quad \langle\Phi\rangle = -E \tag{25}$$

- if one removes energy, the system would be more tightly bound and E would be more negative
- as a consequence, the particles would need to speed up and the **temperature increases**

question

imagine particles in a system would be bound by a harmonic potential $\Phi \propto r^2$. would this system have positive or negative specific heat?

gravothermal instability: particles



globular cluster Omega Centauri, source: Loke Kun Tan

- kinetic energy of a star fluctuates, can get gravitationally unbound
- star leaves cluster on parabolic orbit, does *not* take away energy
- gravitational binding energy distributed among fewer stars
- **system heats up by evaporating stars**, eventually **disintegrates**

question

when does this process stop? what's the final state?

gravothermal instability: particles

- for a gravitationally bound system, we would write $E = -\langle\Phi\rangle$ with the potential energy $\Phi = GM^2/R$
- in the evaporation process, the total energy is approximately conserved, so

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 = \frac{2GM}{R}\frac{\mathrm{d}M}{\mathrm{d}t} - \frac{GM^2}{R^2}\frac{\mathrm{d}R}{\mathrm{d}t} \quad\rightarrow\quad \frac{2R}{M}\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t} \tag{26}$$

- let's assume a simple law for the mass loss:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{M}{\tau} \tag{27}$$

which leads to a decaying exponential $M(t) = M_0 \exp(-t/\tau)$

question

can you combine these equations for a differential equation for $R(t)$ and solve it? what makes it consistent with the $M(t)$-solution?

summary

- the large-scale distribution of matter in the universe forms by gravitational instability
- described by continuity equation, Euler-equation (dark matter is collisionless) and Poisson equation (Newtonian gravity)
- linearisation $\delta \ll 1 \rightarrow$ growth equation
  - growth is homogeneous
  - conserves all statistical properties of the field, especially Gaussianity
- nonlinear regime $\delta \gg 1$: perturbation theory or direct simulation
  - linearisation fails
  - growth becomes inhomogeneous
  - Gaussianity is violated by mode coupling
- galaxy rotation is explained by tidal interaction
- haloes form by gravitational collapse, but their stability is difficult to understand