cosmological large-scale structure

cosmology lecture (chapter 10)

Markus Pössel + Björn Malte Schäfer

Haus der Astronomy and Centre for Astronomy
Fakultät für Physik und Astronomie, Universität Heidelberg

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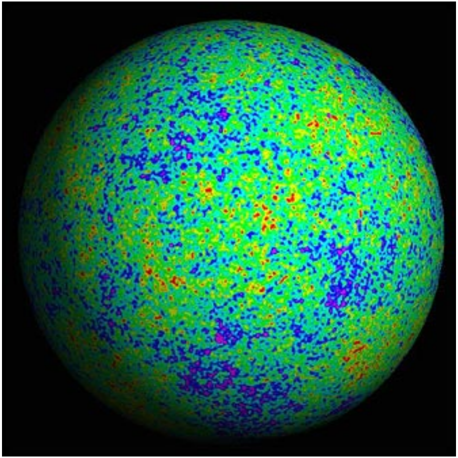
outline

repetition

- flat Friedmann-Lemaître cosmologies with matter and a cosmological constant
- thermal history: big bang nucleosynthesis and formation of atoms
- inflation: solution to flatness and horizon problems
- generation of fluctuations in the distribution of matter
  - quantum fluctuations of the inflaton field perturb gravitational field
  - matter and radiation react on the perturbed gravitational field
- fluctuations of the cosmic microwave background
  - at the time of (re)combination of hydrogen atoms
  - temperature of photons depends on motion and potential depth
  - potential fluctuations are the inflationary perturbations
  - gravitational redshift (Sachs-Wolfe effect) and Doppler shifts in photon temperature
- inflationary perturbations are Gaussian, consequence of the central limit theorem

inflationary fluctuations in the CMB



source: WMAP

random processes

- inflation generates fluctuations in the distribution of matter
  - fluctuations can be seen in the cosmic microwave background
  - seed fluctuations for the large-scale distribution of galaxies
  - amplified by self-gravity
- **cosmology is a statistical subject**
- fluctuations form a **Gaussian random field**
- random processes: specify
  - probability density $p(x)\mathrm{d}x$
  - covariance, in the case of multivariate processes $p(\vec{x})\mathrm{d}\vec{x}$
- measurement of $p(x)\mathrm{d}x$ by determining moments $\langle x^n \rangle = \int \mathrm{d}x\, x^n p(x)$
- cosmology: random process describes the fluctuations of the **overdensity**

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}} \tag{1}$$

with the mean density $\bar{\rho} = \Omega_m \rho_{\mathrm{crit}}$

double pendulum

- **simple example of a random process**
- double pendulum is a chaotic system, dynamics depends **very** sensitively on tiny changes in the initial condition
- random process: imagine you want to know the distribution of φ one minute after starting
  - move to initial conditions and let go
  - wait 1 minute and measure φ (one realisation)
  - repeat experiment → distribution $p(\varphi)\mathrm{d}\varphi$ (ensemble of realisations)
- 2 more types of data
  - distributions and moments of more than one observable
  - moments of observables across different times

question

write down the Lagrangian, perform variation and derive the equation of motion! show that there is a nonlinearity

double pendulum: ergodicity and homogeneity

ergodicity

with time, the dynamics generates values for the observables with the same probability as in the statistical ensemble,
$p(\varphi(t))\mathrm{d}t \propto p(\varphi)\mathrm{d}\varphi$

- time averaging = ensemble averaging, for measuring moments

homogeneity

statistical properties are invariant under time-shifts Δt
$p(\varphi(t))\mathrm{d}\varphi = p(\varphi(t+\Delta t))\mathrm{d}\varphi$

- necessary condition for ergodicity
- double pendulum: not applicable if there is dissipation

Gaussian random fields in cosmology

- fluctuations in the density field are a Gaussian random process → sufficient to measure the **variance**
  - ergodicity: postulated (theorem by Adler)
  - volume averages are equivalent to ensemble averages

$$\langle\delta^n\rangle = \frac{1}{V}\int_V \mathrm{d}^3x\,\delta^n(\vec{x})p(\delta(\vec{x})) \tag{2}$$

  - homogeneity: statistical properties independent of position $\vec{x}$

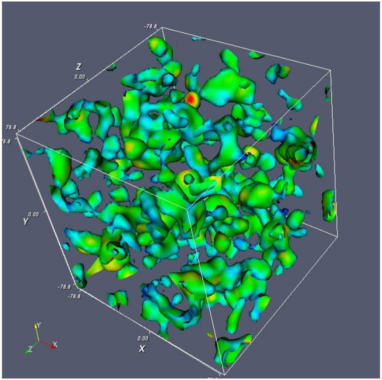
$$p(\delta(\vec{x})) \propto p(\delta(\vec{x}+\Delta\vec{x})) \tag{3}$$

- the density field is a 3d random field → **isotropy**

$$p(\delta(\vec{x})) = p(\delta(R\vec{x})), \text{for all rotation matrices } R \tag{4}$$

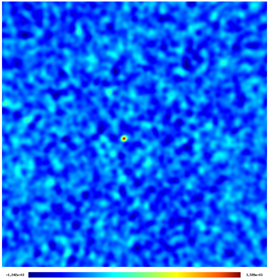
- finite correlation length: amplitudes of δ at two positions $\vec{x}_1$ and $\vec{x}_2$ are not independent:
  - covariance needed for Gaussian distribution $p(\delta(\vec{x}_1),\delta(\vec{x}_2))$
  - measurement of cross variance/covariance $\langle\delta(\vec{x}_1)\delta(\vec{x}_2)\rangle$
  - $\langle\delta(\vec{x}_1)\delta(\vec{x}_2)\rangle$ is called **correlation function** ξ

Gaussian random field

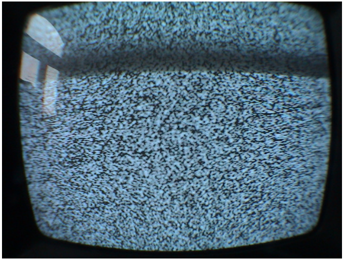


isodensity surfaces, threshold 2.5σ, shading $\sim$ local curvature, CDM power spectrum, smoothed on $8\ \mathrm{Mpc}/h$-scales

statistics: correlation function and spectrum



finite correlation length



zero correlation length

correlation function

quantification of fluctuations: correlation function $\xi(\vec{r}) = \langle\delta(\vec{x}_1)\delta(\vec{x}_2)\rangle$, $\vec{r} = \vec{x}_2 - \vec{x}_1$ for Gaussian, homogeneous fluctuations, $\xi(\vec{r}) = \xi(r)$ for isotropic fields

statistics: correlation function and spectrum

- Fourier transform of the density field

$$\delta(\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\delta(\vec{k})\exp(\mathrm{i}\vec{k}\vec{x}) \leftrightarrow \delta(\vec{k}) = \int \mathrm{d}^3x\,\delta(\vec{x})\exp(-\mathrm{i}\vec{k}\vec{x}) \tag{5}$$

- variance $\langle\delta(\vec{k}_1)\delta^*(\vec{k}_2)\rangle$: use homogeneity $\vec{x}_2 = \vec{x}_1 + \vec{r}$ and $\mathrm{d}^3x_2 = \mathrm{d}^3r$

$$\langle\delta(\vec{k}_1)\delta^*(\vec{k}_2)\rangle = \int \mathrm{d}^3r\,\langle\delta(\vec{x}_1)\delta(\vec{x}_1+\vec{r})\rangle\exp(-\mathrm{i}\vec{k}_2\vec{r})(2\pi)^3\delta_D(\vec{k}_1-\vec{k}_2) \tag{6}$$

  - definition spectrum $P(\vec{k}) = \int \mathrm{d}^3r\,\langle\delta(\vec{x}_1)\delta(\vec{x}_1+\vec{r})\rangle\exp(-\mathrm{i}\vec{k}\vec{r})$
  - spectrum $P(\vec{k})$ is the Fourier transform of the correlation function $\xi(\vec{r})$
  - homogeneous fields: Fourier modes are mutually uncorrelated
  - isotropic fields: $P(\vec{k}) = P(k)$

question

show that the unit of the spectrum $P(k)$ is L^3! what's the relation between $\xi(r)$ and $P(k)$ in an isotropic field?

Gaussianity and the characteristic function

- for a continuous pdf, all moments need to be known for reconstructing the pdf
- reconstruction via **characteristic function** $\phi(t)$ (Fourier transform)

$$\phi(t) = \int \mathrm{d}x p(x) \exp(\mathrm{i}tx) = \int \mathrm{d}x p(x) \sum_n \frac{(\mathrm{i}tx)^n}{n!} = \sum_n \langle x^n \rangle_p \frac{(\mathrm{i}t)^n}{n!}$$

with moments $\langle x^n \rangle = \int \mathrm{d}x x^n p(x)$

- Gaussian pdf is special:
  - all moments exist! (counter example: Cauchy pdf)
  - all even moments are expressible as products of the variance
  - σ is enough to statistically reconstruct the pdf
  - pdf can be differentiated arbitrarily often (Hermite polynomials)

question

show that for a Gaussian pdf $\langle x^{2n} \rangle \propto \langle x^2 \rangle^n$. what's $\phi(t)$?

moment generating function

- variance σ^2 characterises a Gaussian pdf completely
- $\langle x^{2n}\rangle \propto \langle x^2\rangle^n$, but what is the constant of proportionality?
- look at the **moment generating function**

$$M(t) = \int \mathrm{d}x p(x)\exp(tx) = \langle \exp(tx)\rangle_p = \sum_n \langle x^n\rangle_p \frac{t^n}{n!}$$

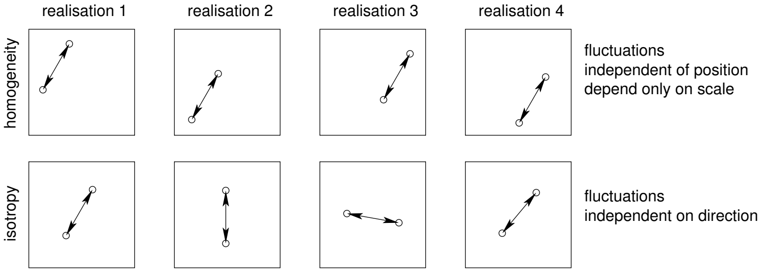
- $M(t)$ is the Laplace transform of pdf $p(x)$, and $\phi(t)$ is the Fourier transform
- nth derivative at $t = 0$ gives moment $\langle x^n\rangle_p$:

$$M'(t) = \langle x\exp(tx)\rangle_p = \langle x\rangle_p$$

question

compute $\langle x^n\rangle$, $n = 2, 3, 4, 5, 6$ for a Gaussian directly (by induction) and with the moment generating function $M(t)$

homegeneity and isotropy in $\xi(r)$



isotropy and homogeneity in an ensemble

- homogeneity: a measurement of $\langle\delta(\vec{x})\delta(\vec{x}+\vec{r})\rangle$ is independent of $\vec{x}$, if one averages over ensembles
- isotropy: a measurement of $\langle\delta(\vec{x})\delta(\vec{x}+\vec{r})\rangle$ does not depend on the direction of $\vec{r}$, in the ensemble averaging

why correlation functions?

a proof for climate change and global warming

please be careful: we measure the correlation function because it characterises the random process generating a realisation of the density field, not because there is a badly understood mechanism relating amplitudes at different points!
(PS: don't extrapolate to 2014!)

tests of Gaussianity

Gaussianity

all moments needed for reconstructing the probability density

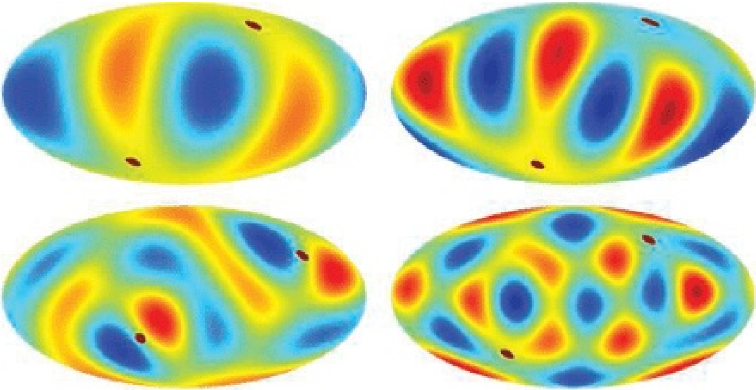
- data is finite: only a limited number of estimators are available
- classical counter example: **Cauchy-distribution**

$$p(x)\mathrm{d}x \propto \frac{\mathrm{d}x}{x^2 + a^2} \tag{7}$$

$\rightarrow$ all even moments are infinite

- genus statistics: peak density, length of isocontours
- independency of Fourier modes

tests of Gaussianity: axis of evil



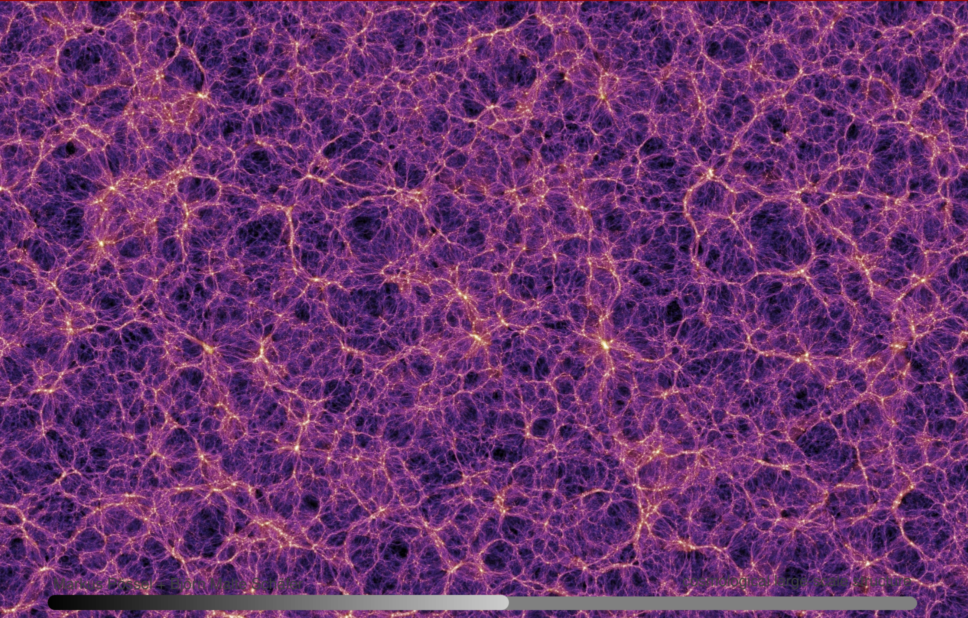
CMB axis of evil: multipole alignment

- CMB-sky: weird (unprobable) alignment between low multipoles

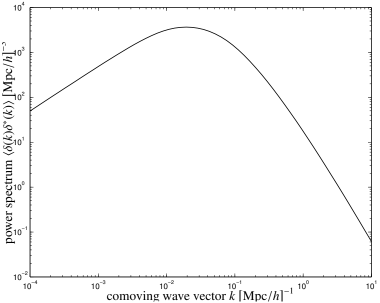
weak and strong Gaussianity

- differentiate weak and strong Gaussianity
- strong Gaussianity: Gaussian distributed amplitudes of Fourier modes
  - implies Gaussian amplitude distribution in real space
  - argumentation: via cumulants
- weak Gaussianity: central limit theorem
  - assume independent Fourier modes, but arbitrary amplitude distribution in Fourier space
  - Fourier transform: many elementary waves contribute to amplitude at a given point
  - central limit theorem: sum over a large number of independent random numbers is Gaussian distributed
  - field in real space is approximately Gaussian, even though the Fourier modes can be arbitrarily distributed

the cosmic web (Millenium simulation)

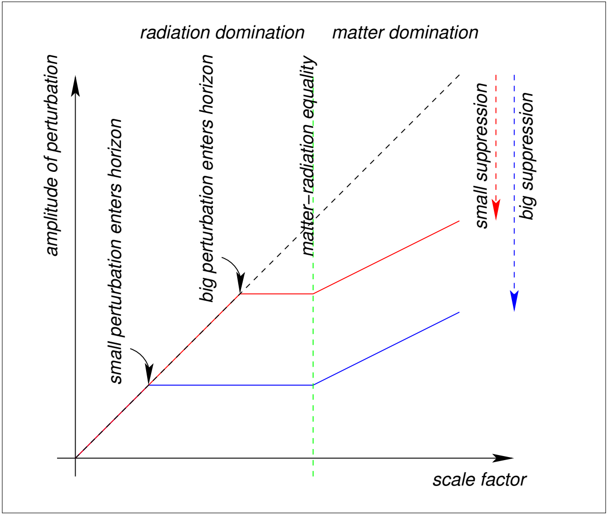


CDM spectrum $P(k)$ and the transfer function $T(k)$



- ansatz for the CDM power spectrum: $P(k) = k^{n_s} T(k)^2$
- small scales suppressed by radiation driven expansion → **Meszaros-effect**
- asymptotics: $P(k) \propto k$ on large scales, and $\propto k^{-3}$ on small scales

Meszaros effect 1



Meszaros effect 2

- perturbation grows $\propto a^2$ outside horizon in the radiation-dominated era (really difficult to understand, need covariant perturbation theory)
- when entering the horizon, fast radiation driven expansion keeps perturbation from growing, dynamical time-scale $t_\mathrm{dyn} \gg t_\mathrm{Hubble} = 1/H(a)$
- all perturbations start growing at the time of matter-radiation equality ($z \simeq 7000$, $\Omega_M(z) = \Omega_R(z)$), growth $\propto a$
- size of the perturbation corresponds to scale factor of the universe at horizon entry
- total suppression is $\propto k^{-2}$, power spectrum $\propto k^{-4}$
- exact solution of the problem: numerical solution for transfer function $T(k)$, with shape parameter Γ, which reflects the matter density

CDM shape parameter Γ

- exact shape of $T(k)$ follows from Boltzmann codes
- express wave-vector k in units of the shape parameter:

$$q \equiv \frac{k/\mathrm{Mpc}^{-1}h}{\Gamma} \tag{8}$$

- Bardeen-fitting formula for low-Ω_m cosmologies

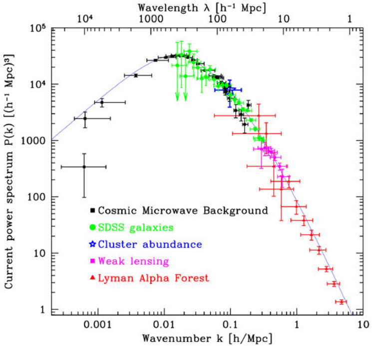
$$T(q) = \frac{\ln(1+eq)}{eq} \times \left[1 + aq + (bq)^2 + (cq)^3 + (dq)^4\right]^{-\frac{1}{4}},$$

- to good approximation $\Gamma = \Omega_m h$
- small $\Gamma \rightarrow$ skewed to left, big $\Gamma \rightarrow$ skewed to right

question

verify the asymptotic behaviour of $T(q)$ for $q \ll 1$ and $q \gg 1$

observational constraints on $P(k)$



data for $P(k)$ from observational probes

- many observational channels are sensitive to $P(k)$
- amazing agreement for the shape

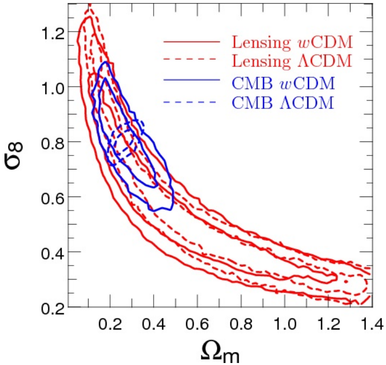
normalisation of the spectrum: σ_8

- CDM power spectrum $P(k)$ needs to be normalised
- observations: fluctuations in the galaxy counts on $8\ \mathrm{Mpc}/h$-scales are approximately constant and $\simeq 1$ (Peebles)
- introduced filter function $W(\vec{x})$
- convolve density field $\delta(\vec{x})$ with filter function $W(\vec{x})$ in real space $\rightarrow$ multiply density field $\delta(\vec{k})$ with filter function $W(\vec{k})$ in Fourier space
- convention: σ_8, $R = 8\ \mathrm{Mpc}/h$

$$\sigma_8^2 = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^2 P(k) W^2(kR) \tag{9}$$

  with a spherical top-hat filter $W(kR)$
- least accurate cosmological parameter, discrepancy between WMAP, lensing and clusters

lensing and CMB constraints on σ_8



constraints on Ω_m and σ_8

- some tension between best-fit values
- possibly related to measurement of galaxy shapes in lensing

cosmological standard model

cosmology + structure formation are described by:

- dark energy density Ω_φ
- cold dark matter density Ω_m
- baryon density Ω_b
- dark energy density equation of state parameter w
- Hubble parameter h
- primordial slope of the CDM spectrum n_s, from inflation
- normalisation of the CDM spectrum σ_8

cosmological standard model: 7 parameters

known to few percent accuracy, amazing predictive power

properties of dark matter

current paradigm:

structures from by gravitational instability from inflationary fluctuations in the cold dark matter (CDM) distribution

- collisionless → very small interaction cross-section
- cold → negligible thermal motion at decoupling, no cut-off in the spectrum $P(k)$ on a scale corresponding to the diffusion scale
- dark → no interaction with photons, possible weak interaction
- matter → gravitationally interacting

main conceptual difficulties

- collisionlessness → hydrodynamics, no pressure or viscosity
- non-saturating interaction (gravity) → extensivity of binding energy

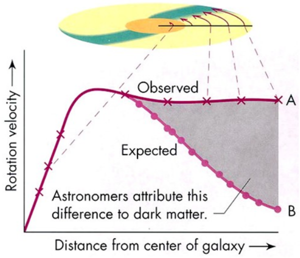
dark matter and the microwave background

- fluctuations in the density field at the time of decoupling are $\simeq 10^{-5}$
- long-wavelength fluctuations grow proportionally to a
- if the CMB was generated at $a = 10^{-3}$, the fluctuations can only be 10^{-2} today
- large, supercluster-scale objects have $\delta \simeq 1$

cold dark matter

need for a **nonbaryonic** matter component, which is not interacting with photons

galaxy rotation curves

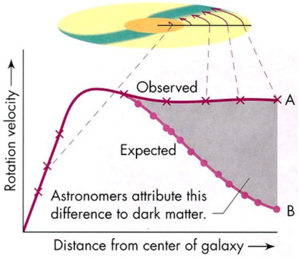


- balance centrifugal and gravitational force
- difficulty: measured in low-surface brightness galaxies
- galactic disk is embedded into a larger halo composed of CDM

question

show that the density profile of a galaxy needs to be $\rho \propto 1/r^2$

galaxy rotation curves



question

realistic haloes are described by the NFW-profile, with 3 regions $\rho \propto 1/r^{\alpha}$ with $\alpha = 1, 2, 3$. can you drive the circular velocity-radius relation in all three regimes?

structure formation equations

cosmic structure formation

cosmic structures are generated from tiny inflationary seed fluctuations, as a fluid mechanical, self-gravitating phenomenon (with Newtonian gravity), on an expanding background

- continuity equation: no matter ist lost or generated

$$\frac{\partial}{\partial t}\rho + \mathrm{div}(\rho\vec{\upsilon}) = 0 \tag{10}$$

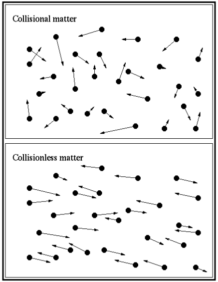
- Euler-equation: evolution of velocity field due to gravitational forces

$$\frac{\partial}{\partial t}\vec{\upsilon} + \vec{\upsilon}\nabla\vec{\upsilon} = -\nabla\Phi \tag{11}$$

- Poisson-equation: potential is sourced by the density field

$$\Delta\Phi = 4\pi G\rho \tag{12}$$

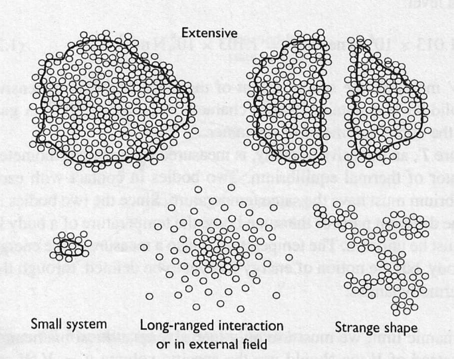
collisionlessness of dark matter



source: P.M. Ricker

- CDM is collisionless (elastic collision cross section $\ll$ neutrinos)
  - why can galaxies rotate and how is vorticity generated?
  - why do galaxies form from their initial conditions without viscosity?
  - how can one stabilise galaxies against gravity without pressure?
  - is it possible to define a temperature of a dark matter system?

non-extensivity of gravity



source: Kerson Huang, statistical physics

- gravitational interaction is long-reached
- gravitational binding energy per particle is not constant for $n \to \infty$

summary

- inflation generates seed fluctuations in the (dark) matter distribution
- fluctuations form a **Gaussian random field**
- description with power spectrum $P(k)$ or correlation function $\xi(r)$
- shape of $P(k)$:
  - inflation: **Harrison-Zel'dovich** spectrum $P(k) \propto k^{n_s}$
  - transition from radiation to matter dominated phase: **transfer function** changes $P(k) \propto k^{n_s} T^2(k)$
  - normalisation: fixed by variance σ_8 on 8 Mpc$/h$ scales
- structures grow by self-gravity:
  - collisionlessness
  - non-extensivity of gravity