cosmological large-scale structure

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flat Friedmann-Lemaître cosmologies with matter and a cosmological constant

thermal history: big bang nucleosynthesis and formation of atoms

inflation: solution to flatness and horizon problems

generation of fluctuations in the distribution of matter
  
  quantum fluctuations of the inflaton field perturb gravitational field
  
  matter and radiation react on the perturbed gravitational field

fluctuations of the cosmic microwave background
  
  at the time of (re)combination of hydrogen atoms
  
  temperature of photons depends on motion and potential depth
  
  potential fluctuations are the inflationary perturbations
  
  gravitational redshift (Sachs-Wolfe effect) and Doppler shifts in photon temperature

inflationary perturbations are Gaussian, consequence of the central limit theorem
inflationary fluctuations in the CMB

source: WMAP
random processes

- inflation generates fluctuations in the distribution of matter
  - fluctuations can be seen in the cosmic microwave background
  - seed fluctuations for the large-scale distribution of galaxies
  - amplified by self-gravity

- cosmology is a statistical subject
- fluctuations form a **Gaussian random field**
- random processes: specify
  - probability density \( p(x)dx \)
  - covariance, in the case of multivariate processes \( p(\vec{x})d\vec{x} \)
- measurement of \( p(x)dx \) by determining moments \( \langle x^n \rangle = \int dx x^n p(x) \)
- cosmology: random process describes the fluctuations of the **overdensity**

\[
\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}
\]  

with the mean density \( \bar{\rho} = \Omega_m \rho_{\text{crit}} \)
double pendulum

- **simple example of a random process**
- double pendulum is a chaotic system, dynamics depends **very** sensitively on tiny changes in the initial condition
- random process: imagine you want to know the distribution of $\varphi$ one minute after starting
  - move to initial conditions and let go
  - wait 1 minute and measure $\varphi$ (one realisation)
  - repeat experiment $\rightarrow$ distribution $p(\varphi)d\varphi$ (ensemble of realisations)
- 2 more types of data
  - distributions and moments of more than one observable
  - moments of observables across different times

**question**

write down the Lagrangian, perform variation and derive the equation of motion! show that there is a nonlinearity
**double pendulum: ergodicity and homogeneity**

**ergodicity**

with time, the dynamics generates values for the observables with the same probability as in the statistical ensemble,

\[ p(\varphi(t))dt \propto p(\varphi)d\varphi \]

- time averaging = ensemble averaging, for measuring moments

**homogeneity**

statistical properties are invariant under time-shifts \( \Delta t \)

\[ p(\varphi(t))d\varphi = p(\varphi(t + \Delta t))d\varphi \]

- necessary condition for ergodicity
- double pendulum: not applicable if there is dissipation
Gaussian random fields in cosmology

- fluctuations in the density field are a Gaussian random process → sufficient to measure the **variance**
  - ergodicity: postulated (theorem by Adler)
  - volume averages are equivalent to ensemble averages
    \[
    \langle \delta^n \rangle = \frac{1}{V} \int_V d^3x \delta^n(\vec{x}) p(\delta(\vec{x}))
    \] (2)
  - homogeneity: statistical properties independent of position \(\vec{x}\)
    \[
    p(\delta(\vec{x})) \propto p(\delta(\vec{x} + \Delta \vec{x}))
    \] (3)
- the density field is a 3d random field → **isotropy**
  \[
  p(\delta(\vec{x})) = p(\delta(R\vec{x})), \text{ for all rotation matrices } R
  \] (4)
- finite correlation length: amplitudes of \(\delta\) at two positions \(\vec{x}_1\) and \(\vec{x}_2\) are not independent:
  - covariance needed for Gaussian distribution \(p(\delta(\vec{x}_1), \delta(\vec{x}_2))\)
  - measurement of cross variance/covariance \(\langle \delta(\vec{x}_1) \delta(\vec{x}_2) \rangle\)
  - \(\langle \delta(\vec{x}_1) \delta(\vec{x}_2) \rangle\) is called **correlation function** \(\xi\)
Gaussian random field

Isodensity surfaces, threshold $2.5\sigma$, shading $\sim$ local curvature, CDM power spectrum, smoothed on $8 \text{ Mpc}/h$-scales
statistics: correlation function and spectrum

finite correlation length
zero correlation length

correlation function
quantification of fluctuations: correlation function
\( \xi(\vec{r}) = \langle \delta(\vec{x}_1)\delta(\vec{x}_2) \rangle \), \( \vec{r} = \vec{x}_2 - \vec{x}_1 \) for Gaussian, homogeneous fluctuations, \( \xi(\vec{r}) = \xi(r) \) for isotropic fields
statistics: correlation function and spectrum

• Fourier transform of the density field

\[ \delta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \delta(\vec{k}) \exp(i\vec{k}\vec{x}) \leftrightarrow \delta(\vec{k}) = \int d^3x \delta(\vec{x}) \exp(-i\vec{k}\vec{x}) \] (5)

• variance \( \langle \delta(\vec{k}_1)\delta^*(\vec{k}_2) \rangle \): use homogeneity \( \vec{x}_2 = \vec{x}_1 + \vec{r} \) and \( d^3x_2 = d^3r \)

\[ \langle \delta(\vec{k}_1)\delta^*(\vec{k}_2) \rangle = \int d^3r \langle \delta(\vec{x}_1)\delta(\vec{x}_1 + \vec{r}) \rangle \exp(-i\vec{k}_2\vec{r})(2\pi)^3\delta_D(\vec{k}_1 - \vec{k}_2) \] (6)

• definition spectrum \( P(\vec{k}) = \int d^3r \langle \delta(\vec{x}_1)\delta(\vec{x}_1 + \vec{r}) \rangle \exp(-i\vec{k}\vec{r}) \)

• spectrum \( P(\vec{k}) \) is the Fourier transform of the correlation function \( \xi(\vec{r}) \)

• homogeneous fields: Fourier modes are mutually uncorrelated

• isotropic fields: \( P(\vec{k}) = P(k) \)

question

show that the unit of the spectrum \( P(k) \) is \( L^3 \)! what’s the relation between \( \xi(r) \) and \( P(k) \) in an isotropic field?
Gaussianity and the characteristic function

- for a continuous pdf, all moments need to be known for reconstructing the pdf
- reconstruction via **characteristic function** $\phi(t)$ (Fourier transform)

$$
\phi(t) = \int dx p(x) \exp(itx) = \int dx p(x) \sum_n \frac{(it)^n}{n!} = \sum_n \langle x^n \rangle_p \frac{(it)^n}{n!}
$$

with moments $\langle x^n \rangle = \int dx x^n p(x)$

- Gaussian pdf is special:
  - all moments exist! (counter example: Cauchy pdf)
  - all even moments are expressible as products of the variance
  - $\sigma$ is enough to statistically reconstruct the pdf
  - pdf can be differentiated arbitrarily often (Hermite polynomials)

**question**

show that for a Gaussian pdf $\langle x^{2n} \rangle \propto \langle x^2 \rangle^n$. what's $\phi(t)$?
moment generating function

- variance $\sigma^2$ characterises a Gaussian pdf completely
- $\langle x^{2n} \rangle \propto (\langle x^2 \rangle)^n$, but what is the constant of proportionality?
- look at the moment generating function
  \[
  M(t) = \int dx p(x) \exp(tx) = \langle \exp(tx) \rangle_p = \sum_n \langle x^n \rangle_p \frac{t^n}{n!}
  \]

- $M(t)$ is the Laplace transform of pdf $p(x)$, and $\phi(t)$ is the Fourier transform
- $n$th derivative at $t = 0$ gives moment $\langle x^n \rangle_p$:
  \[
  M'(t) = \langle x \exp(tx) \rangle_p = \langle x \rangle_p
  \]

question

compute $\langle x^n \rangle$, $n = 2, 3, 4, 5, 6$ for a Gaussian directly (by induction) and with the moment generating function $M(t)$
homogeneity and isotropy in $\xi(r)$

- **homogeneity:** a measurement of $\langle \delta(\vec{x})\delta(\vec{x} + \vec{r}) \rangle$ is independent of $\vec{x}$, if one averages over ensembles.

- **isotropy:** a measurement of $\langle \delta(\vec{x})\delta(\vec{x} + \vec{r}) \rangle$ does not depend on the direction of $\vec{r}$, in the ensemble averaging.
why correlation functions?

please be careful: we measure the correlation function because it characterises the random process generating a realisation of the density field, not because there is a badly understood mechanism relating amplitudes at different points!

(PS: don’t extrapolate to 2014!)
tests of Gaussianity

Gaussianity

all moments needed for reconstructing the probability density

• data is finite: only a limited number of estimators are available
• classical counter example: Cauchy-distribution

\[ p(x)dx \propto \frac{dx}{x^2 + a^2} \]  

→ all even moments are infinite

• genus statistics: peak density, length of isocontours
• independency of Fourier modes
tests of Gaussianity: axis of evil

- CMB-sky: weird (unprobable) alignment between low multipoles
weak and strong Gaussianity

- differentiate weak and strong Gaussianity

- strong Gaussianity: Gaussian distributed amplitudes of Fourier modes
  - implies Gaussian amplitude distribution in real space
  - argumentation: via cumulants

- weak Gaussianity: central limit theorem
  - assume independent Fourier modes, but arbitrary amplitude distribution in Fourier space
  - Fourier transform: many elementary waves contribute to amplitude at a given point
  - central limit theorem: sum over a large number of independent random numbers is Gaussian distributed
  - field in real space is approximately Gaussian, even though the Fourier modes can be arbitrarily distributed
the cosmic web (Millenium simulation)
CDM spectrum $P(k)$ and the transfer function $T(k)$

- ansatz for the CDM power spectrum: $P(k) = k^{n_s} T(k)^2$
- small scales suppressed by radiation driven expansion → **Meszaros-effect**
- asymptotics: $P(k) \propto k$ on large scales, and $\propto k^{-3}$ on small scales
Meszaros effect 1

[Diagram showing the relationship between perturbation amplitude and scale factor, with markers for small and big perturbations entering the horizon, matter-radiation equality, and suppression phases.]
Meszaros effect 2

- perturbation grows $\propto a^2$ outside horizon in the radiation-dominated era (really difficult to understand, need covariant perturbation theory)

- when entering the horizon, fast radiation driven expansion keeps perturbation from growing, dynamical time-scale:
  $t_{\text{dyn}} \gg t_{\text{Hubble}} = 1/H(a)$

- all perturbations start growing at the time of matter-radiation equality ($z \approx 7000$, $\Omega_M(z) = \Omega_R(z)$), growth $\propto a$

- size of the perturbation corresponds to scale factor of the universe at horizon entry

- total suppression is $\propto k^{-2}$, power spectrum $\propto k^{-4}$

- exact solution of the problem: numerical solution for transfer function $T(k)$, with shape parameter $\Gamma$, which reflects the matter density
CDM shape parameter $\Gamma$

- exact shape of $T(k)$ follows from Boltzmann codes
- express wave-vector $k$ in units of the shape parameter:
  \[ q \equiv \frac{k/\text{Mpc}^{-1}h}{\Gamma} \]  
  \( (8) \)
- Bardeen-fitting formula for low-$\Omega_m$ cosmologies
  \[ T(q) = \frac{\ln(1 + eq)}{eq} \times \left[ 1 + aq + (bq)^2 + (cq)^3 + (dq)^4 \right]^{-\frac{1}{4}}, \]
- to good approximation $\Gamma = \Omega_m h$
- small $\Gamma \to$ skewed to left, big $\Gamma \to$ skewed to right

**question**

verify the asymptotic behaviour of $T(q)$ for $q \ll 1$ and $q \gg 1$
observational constraints on $P(k)$

- many observational channels are sensitive to $P(k)$
- amazing agreement for the shape
normalisation of the spectrum: $\sigma_8$

- CDM power spectrum $P(k)$ needs to be normalised
- observations: fluctuations in the galaxy counts on 8 Mpc/$h$-scales are approximately constant and $\approx 1$ (Peebles)
- introduced filter function $W(\vec{x})$
- convolve density field $\delta(\vec{x})$ with filter function $W(\vec{x})$ in real space $\rightarrow$ multiply density field $\delta(\vec{k})$ with filter function $W(\vec{k})$ in Fourier space
- convention: $\sigma_8, R = 8$ Mpc/$h$

$$\sigma_8^2 = \frac{1}{2\pi^2} \int_0^\infty dk \, k^2 P(k) W^2(kR) \quad (9)$$

with a spherical top-hat filter $W(kR)$

- least accurate cosmological parameter, discrepancy between WMAP, lensing and clusters
lensing and CMB constraints on $\sigma_8$

- some tension between best-fit values
- possibly related to measurement of galaxy shapes in lensing
cosmological standard model

cosmology + structure formation are described by:

- dark energy density $\Omega_\varphi$
- cold dark matter density $\Omega_m$
- baryon density $\Omega_b$
- dark energy density equation of state parameter $w$
- Hubble parameter $h$
- primordial slope of the CDM spectrum $n_s$, from inflation
- normalisation of the CDM spectrum $\sigma_8$

**cosmological standard model: 7 parameters**

known to few percent accuracy, amazing predictive power
properties of dark matter

**current paradigm:**
structures from by gravitational instability from inflationary fluctuations in the cold dark matter (CDM) distribution

- collisionless → very small interaction cross-section
- cold → negligible thermal motion at decoupling, no cut-off in the spectrum $P(k)$ on a scale corresponding to the diffusion scale
- dark → no interaction with photons, possible weak interaction
- matter → gravitationally interacting

main conceptual difficulties
- collisionlessness → hydrodynamics, no pressure or viscosity
- non-saturating interaction (gravity) → extensivity of binding energy
fluctuations in the density field at the time of decoupling are $\approx 10^{-5}$

long-wavelength fluctuations grow proportionally to $a$

if the CMB was generated at $a = 10^{-3}$, the fluctuations can only be $10^{-2}$ today

large, supercluster-scale objects have $\delta \approx 1$

cold dark matter

need for a nonbaryonic matter component, which is not interacting with photons
galaxy rotation curves

- balance centrifugal and gravitational force
- difficulty: measured in low-surface brightness galaxies
- galactic disk is embedded into a larger halo composed of CDM

question

show that the density profile of a galaxy needs to be $\rho \propto 1/r^2$
realistic haloes are described by the NFW-profile, with 3 regions $\rho \propto 1/r^\alpha$ with $\alpha = 1, 2, 3$. Can you drive the circular velocity-radius relation in all three regimes?
cosmic structure formation

Cosmic structures are generated from tiny inflationary seed fluctuations, as a fluid mechanical, self-gravitating phenomenon (with Newtonian gravity), on an expanding background.

- Continuity equation: no matter is lost or generated
  \[
  \frac{\partial}{\partial t} \rho + \text{div}(\rho \vec{v}) = 0 \tag{10}
  \]

- Euler-equation: evolution of velocity field due to gravitational forces
  \[
  \frac{\partial}{\partial t} \vec{v} + \vec{v} \nabla \vec{v} = -\nabla \Phi \tag{11}
  \]

- Poisson-equation: potential is sourced by the density field
  \[
  \Delta \Phi = 4\pi G \rho \tag{12}
  \]
collisionlessness of dark matter

- CDM is collisionless (elastic collision cross section ≪ neutrinos)
  - why can galaxies rotate and how is vorticity generated?
  - why do galaxies form from their initial conditions without viscosity?
  - how can one stabilise galaxies against gravity without pressure?
  - is it possible to define a temperature of a dark matter system?

source: P.M. Ricker
non-extensivity of gravity

- gravitational interaction is long-reached
- gravitational binding energy per particle is not constant for $n \to \infty$

source: Kerson Huang, statistical physics
summary

- inflation generates seed fluctuations in the (dark) matter distribution
- fluctuations form a **Gaussian random field**
- description with power spectrum $P(k)$ or correlation function $\xi(r)$
- shape of $P(k)$:
  - inflation: **Harrison-Zel’dovich** spectrum $P(k) \propto k^{n_s}$
  - transition from radiation to matter dominated phase: **transfer function** changes $P(k) \propto k^{n_s} T^2(k)$
  - normalisation: fixed by variance $\sigma_8$ on 8 Mpc/$h$ scales
- structures grow by self-gravity:
  - collisionlessness
  - non-extensivity of gravity