inflation and the cosmic microwave background

cosmology lecture (chapter 9)

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   ideas behind inflation
   mechanics of inflation

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   parameter sensitivity

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6 summary
• Friedmann-Lemaître cosmologies explain the expansion dynamics $H(a)$ of the universe

• fluids influence Hubble-expansion according to their eos parameter
  • $w > -1/3$: expansion slows down
  • $w < -1/3$: expansion accelerates

• dark energy and the cosmological constant can cause accelerated expansion $q < 0$, for sufficiently negative eos $w > -1/3$

• Hubble expansion is an adiabatic process, temperature drops while expanding, $T(a)$ depends on adiabatic index $\kappa$

• relic radiation (highly redshifted) from early epochs of the universe
  • big bang nucleosynthesis: neutrino background $T_{\nu} = 1.95K$
  • formation of hydrogen atoms: microwave background $T_{\text{CMB}} = 2.75K$
expansion history of the universe

expansion history of the universe
Planck-scale

- at $a = 0$, $z = \infty$ the metric diverges, and $H(a)$ becomes infinite
- description of general relativity breaks down, quantum effects become important
- relevant scales:
  - quantum mechanics: de Broglie-wave length: $\lambda_{QM} = \frac{2\pi\hbar}{mc}$
  - general relativity: Schwarzschild radius: $r_s = \frac{2Gm}{c^2}$
- setting $\lambda_{QM} = r_s$ defines the Planck mass

$$m_P = \sqrt{\frac{\hbar c}{G}} \approx 10^{19} \text{GeV}/c^2$$ (1)

question

how would you define the corresponding Planck length and the Planck time? what are their numerical values?
flatness problem

• construct a universe with matter \( w = 0 \) and curvature \( w = -1/3 \)

• Hubble function

\[
\frac{H^2(a)}{H_0^2} = \frac{\Omega_m}{a^3} + \frac{\Omega_K}{a^2}
\]  \hspace{1cm} (2)

• density parameter associated with curvature

\[
\frac{\Omega_K(a)}{\Omega_K} = \frac{H_0^2}{a^3(1+w)H^2(a)} = \frac{H_0^2}{a^2H^2(a)}
\]  \hspace{1cm} (3)

• \( \Omega_K \) increases always and was smaller in the past

\[
\Omega_K(a) = \left(1 + \frac{\Omega_m}{\Omega_K} \frac{1}{a}\right)^{-1} \approx \frac{\Omega_K}{\Omega_m}a
\]  \hspace{1cm} (4)

• we know (from CMB observations) that curvature is very small today, typical limits are \( \Omega_K < 0.01 \rightarrow \textbf{even smaller in the past} \)
   • at recombination \( \Omega_K \approx 10^{-5} \)
   • at big bang nucleosynthesis \( \Omega_K \approx 10^{-12} \)
horizon problem

- horizon size: light travel distance during the age of the universe

\[ \chi_H = c \int \frac{da}{a^2 H(a)} \]  

(5)

- assume \( \Omega_m = 1 \), integrate from \( a_{\min} = a_{\text{rec}} \ldots a_{\max} = 1 \)

\[ \chi_H = 2 \frac{c}{H_0} \sqrt{\Omega_m a_{\text{rec}}} = 175 \sqrt{\Omega_m} \text{Mpc}/h \]  

(6)

- comoving size of a volume around a point at recombination inside which all points are in causal contact

- angular diameter distance from us to the recombination shell:

\[ d_{\text{rec}} \approx 2 \frac{c}{H_0} a_{\text{rec}} \approx 5\text{Mpc}/h \]  

(7)

- angular size of the particle horizon at recombination: \( \theta_{\text{rec}} \approx 2^\circ \)

- points in the CMB separated by more than \( 2^\circ \) have never been in causal contact \( \rightarrow \) why is the CMB so uniform if there is no possibility of heat exchange?
inflation: phenomenology

- curvature $\Omega_K \propto$ to the **comoving Hubble radius** $c/(aH(a))$
- if by some mechanism, $c/(aH)$ could decrease, it would drive $\Omega_K$ towards 0 and solve the fine-tuning required by the flatness problem
- shrinking comoving Hubble radius:
  \[ \frac{d}{dt} \left( \frac{c}{aH} \right) = -c \frac{\ddot{a}}{\dot{a}^2} < 0 \rightarrow \ddot{a} > 0 \rightarrow q < 0 \] (8)
- equivalent to the notion of accelerated expansion
- accelerated expansion can be generated by a dominating fluid with sufficiently negative equation of state $w = -1/3$
- horizon problem: fast expansion in inflationary era makes the universe grow from a small, causally connected region

**question**

what’s the relation between deceleration $q$ and equation of state $w$?
inflaton-driven expansion

• analogous to dark energy, one postulates an **inflaton field** $\phi$, with a small kinetic and a large potential energy, for having a sufficiently negative equation of state for accelerated expansion

• pressure and energy density of a homogeneous scalar field

\[
p = \frac{\dot{\phi}^2}{2} - V(\phi), \quad \rho = \frac{\dot{\phi}^2}{2} + V(\phi)
\]  

(9)

• Friedmann equation

\[
H^2(a) = \frac{8\pi G}{3} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right)
\]  

(10)

• continuity equation

\[
\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}
\]  

(11)
slow roll conditions

- inflation can only take place if $\dot{\phi}^2 \ll V(\phi)$

- inflation needs to keep going for a sufficiently long time:
  \[
  \frac{d}{dt} \dot{\phi}^2 \ll \frac{d}{dt} V(\phi) \rightarrow \ddot{\phi} \ll \frac{d}{d\phi} V(\phi) \quad (12)
  \]

- in this regime, the Friedmann and continuity equations simplify:
  \[
  H^2 = \frac{8\pi G}{3} V(\phi), \quad 3\dot{H} = -\frac{d}{d\phi} V(\phi) \quad (13)
  \]

- conditions are fulfilled if
  \[
  \frac{1}{24\pi G} \left( \frac{V'}{V} \right)^2 \equiv \epsilon \ll 1, \quad \frac{1}{8\pi G} \left( \frac{V''}{V} \right) \equiv \eta \ll 1 \quad (14)
  \]

- $\epsilon$ and $\eta$ are called \textbf{slow-roll parameters}
stopping inflation

- flatness problem: shrinkage by $\simeq 10^{30} \simeq \exp(60) \rightarrow 60$ e-folds
- due to the slow-roll conditions, the energy density of the inflaton field is almost constant
- all other fluid densities drop by huge amounts, $\rho_m$ by $10^{90}$, $\rho_\gamma$ by $10^{120}$
- eventually, the slow roll conditions are not valid anymore, the effective equation of state becomes less negative, accelerated expansion stops
- **reheating**: couple $\phi$ to other particle fields, and generate particles from the inflaton’s kinetic energy

**question**

the temperature at the beginning of inflation was equal to the Planck-temperature. what is its value (in Kelvin) and by how much has it dropped until today?
generation of fluctuations

- fluctuations of the inflation field can perturb the distribution of all other fluids
- mean fluctuation amplitude is related to the variance of \( \phi \)
- fluctuations in \( \phi \) perturb the metric, and all other fluids feel a perturbed potential
- relevant quantity

\[
\sqrt{\langle \delta \Phi^2 \rangle} \approx \frac{H^2}{V}
\]  

which is approximately constant during slow-roll

- Poisson-equation in Fourier-space \( k^2 \Phi(k) = -\delta(k) \)
- variance of density perturbations:

\[
|\delta(k)|^2 \propto k^4 |\delta \Phi|^2 \propto k^3 P(k)
\]  

- defines **spectrum** \( P(k) \) of the initial fluctuations, \( P(k) \propto k^n \) with \( n \approx 1 \)
- fluctuations are Gaussian, because of the **central limit theorem**
random fields

- random process $\rightarrow$ probability density $p(\delta)d\delta$ of event $\delta$
- alternatively: all moments $\langle \delta^n \rangle = \int d\delta \; \delta^n p(\delta)$
- in cosmology:
  - random events are values of the density field $\delta$
  - outcomes for $\delta(\vec{x})$ form a statistical ensemble at fixed $\vec{x}$
  - ergodic random processes:
    one realisation is consistent with $p(\delta)d\delta$
- special case: Gaussian random field
  - only $\textbf{variance}$ relevant
characteristic function $\phi(t)$

- for a continuous pdf, all moments need to be known for reconstructing the pdf
- reconstruction via **characteristic function** $\phi(t)$ (Fourier transform)

\[
\phi(t) = \int dx p(x) \exp(itx) = \int dx p(x) \sum_n \frac{(itx)^n}{n!} = \sum_n \langle x^n \rangle_p \frac{(it)^n}{n!} \quad (17)
\]

with moments $\langle x^n \rangle = \int dx x^n p(x)$

- Gaussian pdf is special:
  - all moments exist! (counter example: Cauchy pdf)
  - all odd moments vanish
  - all even moments are expressible as products of the variance
  - $\sigma$ is enough to statistically reconstruct the pdf
  - pdf can be differentiated arbitrarily often (Hermite polynomials)
- funky notation: $\phi(t) = \langle \exp(itx) \rangle$
cosmic microwave background

- Inflation has generated perturbations in the distribution of matter.
- The hot baryon plasma feels fluctuations in the distribution of (dark) matter by gravity.
- At the point of (re)combination:
  - Hydrogen atoms are formed.
  - Photons can propagate freely.
- Perturbations can be observed by two effects:
  - Plasma was not at rest, but flowing towards a potential well → Doppler-shift in photon temperature, depending on direction of motion.
  - Plasma was residing in a potential well → gravitational redshift.
- Between the end of inflation and the release of the CMB, the density field was growth **homogeneously** → all statistical properties of the density field are conserved.
- Testing of inflationary scenarios is possible in CMB observations.
formation of hydrogen: (re)combination

- temperature of the fluids drops during Hubble expansion
- eventually, the temperature is sufficiently low to allow the formation of hydrogen atoms
- but: high photon density (remember $\eta_B = 10^9$) can easily reionise hydrogen
- Hubble-expansion does not cool photons fast enough between recombination and reionisation
- neat trick: recombination takes place by a 2-photon process

question
at what temperature would the hydrogen atoms form if they could recombine directly? what redshift would that be?

question
why do we observe a continuum spectrum from the formation of...
CMB motion dipole

- the most important structure on the microwave sky is a dipole
- CMB dipole is interpreted as a relative motion of the earth
- CMB dispole has an amplitude of $10^{-3} K$, and the peculiar velocity is $\beta = 371\,\text{km/s} \cdot c$

$$T(\theta) = T_0 (1 + \beta \cos \theta)$$

**question**

is the Planck-spectrum of the CMB photons conserved in a Lorentz-boost?

**question**

would it be possible to distinguish between a motion dipole and an intrinsic CMB dipole?
CMB dipole

source: COBE
subtraction of motion dipole: primary anisotropies

source: PLANCK simulation
CMB angular spectrum

- analysis of fluctuations on a sphere: decomposition in $Y_{\ell m}$

$$T(\theta) = \sum_{\ell} \sum_{m} t_{\ell m} Y_{\ell m}(\theta) \iff t_{\ell m} = \int d\Omega \ T(\theta) Y_{\ell m}^*(\theta) \quad (19)$$

- spherical harmonics are an orthonormal basis system

- average **fluctuation variance** on a scale $\ell \approx \pi/\theta$

$$C(\ell) = \langle |t_{\ell m}|^2 \rangle \quad (20)$$

- averaging $\langle \ldots \rangle$ is a hypothetical ensemble average. In reality, one computes an estimate of the variance,

$$C(\ell) \approx \frac{1}{2\ell + 1} \sum_{m=-\ell}^{m=+\ell} |t_{\ell m}|^2 \quad (21)$$
standard ruler principle

trinity nuclear test, 16 milli-seconds after explosion

• physical size: combine
  1 time since explosion
  2 velocity of fireball

• distance: combine
  1 physical size
  2 angular size

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formation of baryon acoustic oscillations

- from a pointlike perturbation, a spherical wave travels in the photon-baryon-plasma
- propagation stops when atoms form
cosmic microwave background: standard ruler

- hot and cold patches of the CMB have a typical physical size, related to the horizon size at the time of formation of hydrogen atoms
- idea: physical size and apparent angle are related, redshift of decoupling known
Standard ruler: measurement principle (Eisenstein)

- Curvature can be well constrained
- Assumption: galaxy bias understood, nonlinear structure formation not too important

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parameter sensitivity of the CMB spectrum

source: WMAP

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features in the CMB spectrum

• predicting the spectrum $C(\ell)$ is very complicated
• perturbations in the CMB photons $n \propto T^3$, $u \propto T^4$, $p = u/3 \propto T^4$:

\[
\frac{\delta n}{n_0} = 3 \frac{\delta T}{T} \equiv \Theta, \quad \frac{\delta u}{u_0} = 4\Theta = \frac{\delta p}{p_0}
\]  (22)

• continuity and Euler equations:

\[
\dot{n} = n_0 \text{div} \nu = 0, \quad \dot{\nu} = -c^2 \frac{\nabla p}{u_0 + p_0} + \nabla \delta \Phi
\]  (23)

• use $u_0 + p_0 = 4/3u_0 = 4p_0$

• combine both equations

\[
\ddot{\Theta} - \frac{c^2}{3} \Delta \Theta + \frac{1}{3} \Delta \delta \Phi = 0
\]  (24)

• identify two mechanisms:
  • oscillations may occur, and photons experience Doppler shifts
  • photons feel fluctuations in the potential: Sachs-Wolfe effect
parameter sensitivity of the CMB spectrum

source: Wayne Hu
baryon acoustic oscillations in the galaxies

- pair density $\xi(r)$ of galaxies as a function of separation $r$

- baryon acoustic oscillations: the (pair) density of galaxies is enhanced at a separation of about 100Mpc/$h$ comoving

- idea: angle under which this scale is viewed depends on redshift
secondary CMB anisotropies

- CMB photons can do interactions in the cosmic large-scale structure on their way to us
- two types of interaction: Compton-collisions and gravitational
- consequence: secondary anisotropies
- study of secondaries is very interesting: observation of the growth of structures possible, and precision determination of cosmological parameters
- all effects are in general important on small angular scales below a degree
thermal Sunyaev-Zel’dovich effect

- Compton-interaction of CMB photons with thermal electrons in clusters of galaxies
- characteristic redistribution of photons in energy spectrum
**kinetic Sunyaev-Zel’dovich/Ostriker-Vishniac effect**

- Compton-interaction of CMB photons with electrons in **bulk flows**
- Increase/decrease in CMB temperature according to direction of motion

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**thermal SZ sky map**

**CMB spectrum distortion**

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CMB lensing

- gravitational deflection of CMB photons on potentials in the cosmic large-scale structure
- CMB spots get distorted, and their fluctuation statistics is changed, in particular the polarisation

source: A. Lewis, A. Challinor
integrated Sachs-Wolfe effect

- gravitational interaction of photons with time-evolving potentials
- higher-order effect on photon geodesics in general relativity

source: B. Barreiro
summary

- inflation:
  - solves flatness-problem
  - solves horizon-problem
  - generates perturbations in the density field

- perturbations are Gaussian, and can be described by a correlation function $\xi(r)$ or a power spectrum $P(k)$

- perturbations have a scale-free spectrum $P(k) \propto k^{n_s}$ with $n_s \approx 1$

- Meszaros-effect changes spectrum to $P(k) \propto k^{n_s-4}$ on small scales

- normalisation of the CDM spectrum $P(k)$ by the parameter $\sigma_8$

- cosmic microwave background probes inflationary perturbations
  - dynamics of the plasma in potential fluctuations
  - precision determination of cosmological parameters
  - secondary effects influence the CMB on small scales (gravitational lensing, Sunyaev-Zel'dovich effect, integrated Sachs-Wolfe effect)