Basic cosmology

Heraeus Summer School 2013

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Contents

The framework behind cosmology

Modern framework for cosmology: **homogeneous, isotropic models based on general relativity**

- A surprising amount can be derived in a Newtonian way! (cf. tutorial)
- Inaccessible to Newton:
  - consistent theory of the behavior of light
  - non-flat spaces
  - some sources of gravity

$\Rightarrow$ general relativity needed!

Goals of this lecture

- Introduce Friedmann-Lemaître-Robertson-Walker models
- Distinguish between kinematic (scale factor, metric) and dynamic (Friedmann equations) effects
- Introduce concepts, motivation and nomenclature needed in later lectures

General relativity

- Einstein's theory (1915) linking space-time-geometry and gravity
- Central model-building element: space-time metric $g_{\mu\nu}$ (or line element $\mathrm{d}s^2$)
- Light and freely falling particles move along the straightest-possible space-time paths (geodesics)
- Einstein's equations link space-time geometry ("Einstein tensor $G_{\mu\nu}$") and sources of gravity ("Stress-energy tensor $T_{\mu\nu}$)
- Physical models: solutions of Einstein's equations

...for our purposes:

- What does it mean to have a universal scale factor?
- Heuristic introduction of the metric
- (Newtonian) derivation of dynamic equations ($\Rightarrow$ tutorial)
- Basic parameters
- Observing the expanding universe

Expansion with a scale factor

Deduced from Hubble's (and later) results:

scale factor expansion

Idea: **cosmic substratum** (i.e. "galaxy dust") floating in space.

All distances between substratum particles change proportional to $a(t)$, e.g. distance d_{12} between galaxy 1 and galaxy 2:

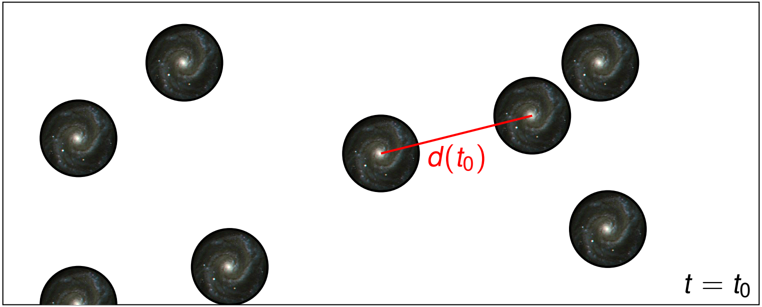
$$d_{12}(t_1) = \frac{a(t_1)}{a(t_0)} \cdot d_{12}(t_0).$$

Expansion with a scale factor

Expansion with a scale factor

Distances between galaxies

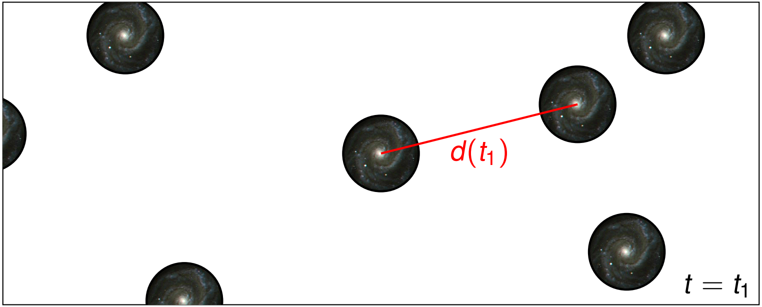
Consider galaxies in the Hubble flow:



All distances change as $d(t) = \frac{a(t)}{a(t_0)} \cdot d(t_0)$.

Distances between galaxies

Consider galaxies in the Hubble flow:



All distances change as $d(t) = \dfrac{a(t)}{a(t_0)} \cdot d(t_0)$.

Expansion with a scale factor

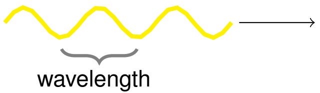
Signals emitted at time t_e with time difference Δt_e, initial spatial separation $v \cdot \Delta t$

$\Rightarrow$ if separation is stretched with the expansion factor and signals arrive at time t_1, then

$$\Delta t_0 = \frac{a(t_0)}{a(t_e)} \cdot \Delta t_e.$$

Cosmological redshift

Wavelength scaling with scale factor:



Redshift defined as

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(t_0)}{a(t_e)} - 1$$

or

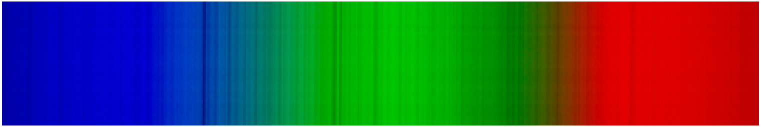
$$1 + z = \frac{a(t_0)}{a(t_e)}$$

For co-moving galaxies: z is directly related to r_e.

For monotonous $a(t)$: distance measure.

Cosmological redshift

Redshift for $a(t_0) > a(t_e)$; blueshift for $a(t_0) < a(t_e)$



Taylor expansion of the scale factor

Generic Taylor expansion:

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a} \cdot (t - t_0)^2 + \ldots$$

Re-define the Taylor parameters by introducing two functions

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \quad \text{and} \quad q(t) \equiv -\frac{\ddot{a}(t)a(t)}{\dot{a}(t)^2}$$

and corresponding constants

$$H_0 \equiv H(t_0) \quad \text{and} \quad q_0 \equiv q(t_0)$$

$$a(t) = a_0\left[1 + (t - t_0)H_0 - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + \ldots\right]$$

Some nomenclature and values 1/2

t_0 is the standard symbol for the **present time**. If coordinates are chosen so cosmic time $t = 0$ denotes the time of the big bang (phase), then t_0 is the **age of the universe**. Sometimes, the age of the universe is denoted by τ.

$H(t)$ is the **Hubble parameter** (sometimes misleadingly *Hubble constant*)

$H_0 \equiv H(t_0)$ is the **Hubble constant**. Current values around

$$H_0 = 70\,\frac{\mathrm{km/s}}{\mathrm{Mpc}}.$$

Some nomenclature and values 2/2

Sometimes, the Hubble constant is written as

$$H_0 = h \cdot 100\ \frac{\text{km/s}}{\text{Mpc}}$$

(keep your options open!) with h the **dimensionless Hubble constant**.

The inverse of the Hubble constant is the **Hubble time** (cf. the linear case and the models later on).

$$\frac{1}{h \cdot 100\frac{\text{km/s}}{\text{Mpc}}} \approx h^{-1} \cdot 10^{10}\ \text{a}.$$

For “nearby” galaxies. . .

. . . use the Taylor expansion

$a(t) = a(t_0)[1 + H_0(t - t_0) + O((t - t_0)^2)]$:

$$1 - z \approx \frac{1}{1+z} = \frac{a(t_e)}{a(t_0)} \approx 1 + H_0(t_e - t_0)$$

or, using the classical Doppler formula $v = cz$ and defining distance $d = c\Delta t$ for light-travel time Δt,

$$v = cz \approx H_0\, c(t_0 - t_e) \approx H_0 d$$

for small z, small $t_0 - t_e$.

This is Hubble’s law.

Originally discovered by Alexander Friedmann (cf. Stigler’s law).

The Friedmann-Lemaître-Robertson-Walker metric

Next step: Re-visit scale factor expansion within the framework of general relativity.

Space-time described by a *metric*

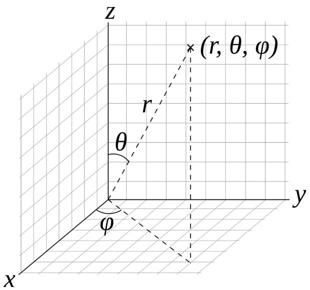
. . . say what?

The meaning of the metric: the flat/Cartesian case

Ordinary Cartesian coordinates: Pythagoras!

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$$

The meaning of the metric: spherical coordinates



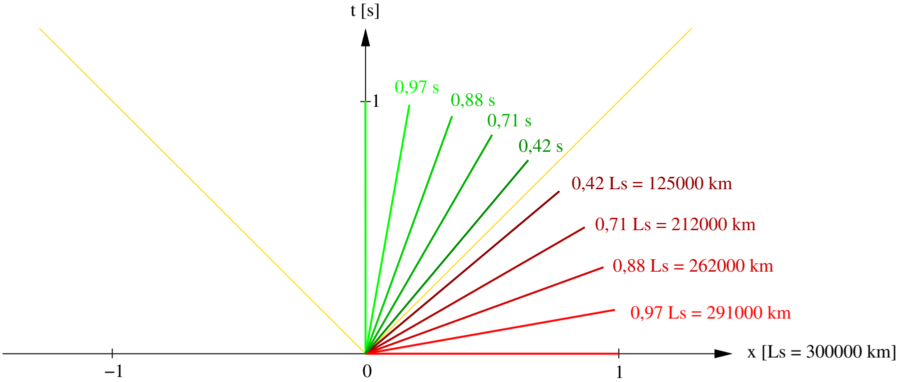
Introduce spherical coordinates r, θ, ϕ:

$$\begin{aligned} x &= r \cdot \sin\theta \cdot \cos\phi \\ y &= r \cdot \sin\theta \cdot \sin\phi \\ z &= r \cdot \cos\theta \end{aligned}$$

The line element is given by

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \equiv dr^2 + r^2 d\Omega.$$

A different kind of metric: Special relativity



$$ds^2 = -c^2 d\tau^2 = d\vec{x}^2 - c^2 dt^2.$$

Cosmological model-building: strategy

Two-step model-building strategy guided by the cosmological principle:

1. Build idealized exactly homogeneous and isotropic models: **Friedmann-Lemaître-Robertson-Walker, FLRW** (exact family of solutions; this lecture)
2. Add inhomogeneities on smaller scales as perturbations (Björn Malte Schäfer on Saturday)

The Friedmann-Robertson-Walker Metric

Only possibility of describing an *isotropic* and *homogeneous* universe (up to freedom of choosing different coordinates):

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega \right]$$

with $a(t)$ the **cosmic scale factor**. From now on: convention $c \equiv 1$.

K a parameter that describes spatial geometry:

$$K = \begin{cases} +1 & \text{spherical space} \\ 0 & \text{Euclidean space} \\ -1 & \text{hyperbolical space} \end{cases}$$

A caveat: global vs. local

Globally, there is *topology* to consider — e.g. a flat metric can belong to infinite Euclidean space, but also, say, to a torus (a patch of Euclidean space with certain identifications).

- $K = 0$: 18 topologically different forms of space. Some inifinite, some finite.
- $K = +1$: inifinitely many topologically different forms. All are finite.
- $K = -1$: infinitely many topologically different forms of space. Some inifinite, some finite.

Light in an FRW universe

Great advantage of GR: Light propagation along geodesic (straightest-possible) lines!

As in special relativity: for light, $\mathrm{d}s^2 = 0$ (alternative: geodetic equation).

Also: use symmetries! Move origin of your coordinate system wherever convenient. Look only at radial movement.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega\right]$$

becomes

$$\mathrm{d}t = \pm\frac{a(t)\,\mathrm{d}r}{\sqrt{1-Kr^2}}.$$

$\Rightarrow$ this will become important in Andreas Just's lecture

The evolution of the scale factor

Up to now, all our conclusions drawn from metric — derived by *symmetry*.

To find the explicit form of $a(t)$ in a rigorous way, we need Einstein's equations,

$$G_{\mu\nu} = 8\pi G\ T_{\mu\nu}$$

with a perfect-fluid stress-energy tensor

$$T^{\mu\nu} = (\rho + p)\,u^{\mu}u^{\nu} + pg^{\mu\nu}$$

— both are beyond the scope of this lecture.

Here, we will directly skip to the solution — in the tutorial, we will find sort-of-Newtonian answers

Solving Einstein's equations for FRW

00 component of Einstein's eq.:

$$3\,\frac{\dot{a}^2+K}{a^2} = 8\pi G\,\rho$$

*i*0 components vanish. *ij* components give

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2+K}{a^2} = -8\pi G p$$

These are the **Friedmann equations**. Their solutions are the **Friedmann-Lemaître-Robertson-Walker** (FLRW) universes.

Re-casting the Friedmann equations

$$\frac{\dot{a}^2 + K}{a^2} = \frac{8\pi G\,\rho}{3}$$

and for $\dot{a} \neq 0$
(by differentiating the above and inserting the *ij*-equation)

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) = -3H(t)(\rho + p).$$

(in tutorial, this will be linked to energy conservation).

How does density change with the scale factor?

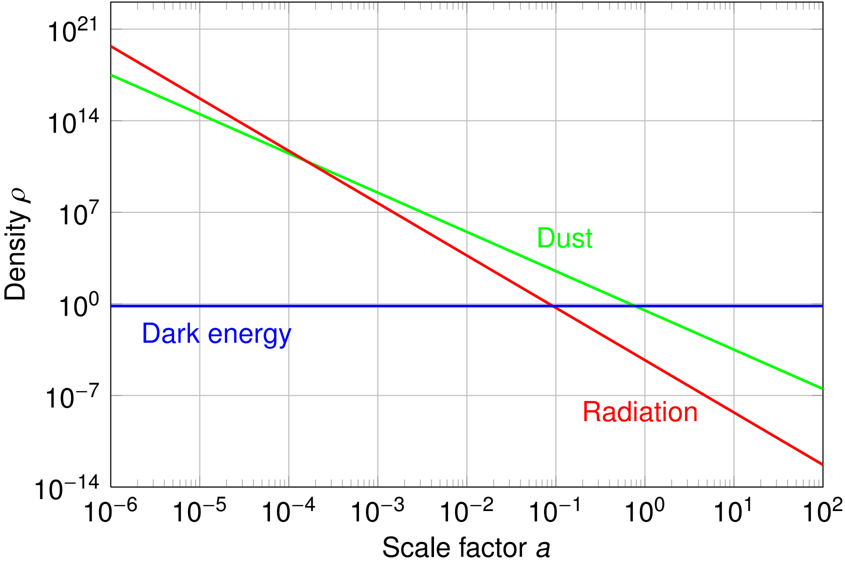
$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) = -3H(t)(\rho + p).$$

links density changes to expansion. Details depend on *equations of state*. In cosmology, most important examples satisfy $p = w\rho$:

1. **Dust**: $w = 0 \quad \Rightarrow \quad \rho \sim 1/a^3$
2. **Radiation**: $w = 1/3 \quad \Rightarrow \quad \rho \sim 1/a^4$
3. **Dark energy/cosmological constant**: $w = -1 \;\; \rho =$const.

Whenever these are the only important components, a universe can have different *phases* — depending on size, different components will dominate.

Different eras depending on the scale factor



Deceleration from the Friedmann equations

Recombine Friedmann equations to give equation for $\ddot{a}$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$\Rightarrow$

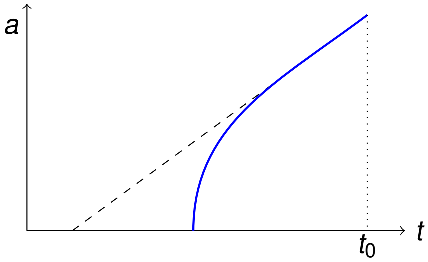
$$q_0 = \frac{4\pi G}{3H_0^2}(\rho_0 + 3p_0)$$

(with ρ_0 and p_0 the present density/pressure).

Recall equation of state $p = w\rho$: Ordinary matter decelerates, but $w = -1$ accelerates via its pressure!

The initial singularity

Shows that, for universes where dark energy ($w = -1$) does not dominate, $\ddot{a}/a \leq 0$. But by our earlier considerations (eras), this is true for small a, that is, in the early universe!



Initial singularity — special case of Hawking-Penrose theorems.

Convenient: Choose as zero point for cosmic time.

The critical density and the Ω parameters

Rescale all present densities (M Matter, R Radiation, Λ dark energy) in terms of the present critical density,

$$\rho_{c0} \equiv \frac{3H_0^2}{8\pi G},$$

and re-scale K accordingly:

$$\Omega_\Lambda = \rho_\lambda(t_0)/\rho_{c0}, \quad \Omega_M = \rho_M(t_0)/\rho_{c0},$$
$$\Omega_R = \rho_R(t_0)/\rho_{c0}, \quad \Omega_K = -K/(a_0 H_0)^2.$$

Re-write the present-day Friedmann equation as

$$\Omega_\Lambda + \Omega_M + \Omega_R + \Omega_K = 1.$$

⇒ link densities and spatial geometry (important later on for CMB)

General considerations for FLRW models

Scaling behaviour of the different densities means that

$$\rho(t) = \frac{3H_0^2}{8\pi G}\left[\Omega_M\left(\frac{a_0}{a(t)}\right)^3 + \Omega_R\left(\frac{a_0}{a(t)}\right)^4 + \Omega_\Lambda\right].$$

Substitute back into the Friedmann equation

$$\frac{\dot{a}^2 + K}{a^2} = \frac{8\pi G\,\rho}{3}$$

and substitute $x(t) \equiv a(t)/a_0 = 1/(1+z)$ to obtain

$$\mathrm{d}t = \frac{\mathrm{d}x}{H_0\, x\,\sqrt{\Omega_\Lambda + \Omega_K\, x^{-2} + \Omega_M\, x^{-3} + \Omega_R\, x^{-4}}}.$$

The age of the universe in FLRW models

Defining $t = 0$ by $a(0) = 0$ [initial singularity] corresponds to $z \to \infty$ or $x = 0$. With this zero point, emission time $t_E(z)$ of light reaching us with redshift z:

$$t_E(z) = \frac{1}{H_0} \int_0^{1/(1+z)} \frac{\mathrm{d}x}{x\,\sqrt{\Omega_\Lambda + \Omega_K\,x^{-2} + \Omega_M\,x^{-3} + \Omega_R\,x^{-4}}}.$$

Special case $z = 0$ corresponds to the present time — gives the *age of the universe* τ,

$$\tau = \frac{1}{H_0} \int_0^{1} \frac{\mathrm{d}x}{x\,\sqrt{\Omega_\Lambda + \Omega_K\,x^{-2} + \Omega_M\,x^{-3} + \Omega_R\,x^{-4}}}.$$

The acceleration parameter q_0

Present pressure:

$$p_0 = \frac{3H_0^2}{8\pi G}(-\Omega_\Lambda + \frac{1}{3}\Omega_R).$$

inserting in

$$q_0 = \frac{4\pi G}{3H_0^2}(\rho_0 + 3p_0),$$

we find that

$$q_0 = \frac{1}{2}(\Omega_M - 2\Omega_\Lambda + 2\Omega_R).$$

The fate of the universe

Rewrite

$$\frac{\dot{a}^2 + K}{a^2} = \frac{8\pi G\rho}{3}$$

as

$$\dot{a}^2 = (H_0 a_0)^2 \left[\Omega_\Lambda x^2 + \Omega_M x^{-1} + \Omega_R x^{-2} + \Omega_K\right]$$

where $x = a/a_0$. Forget about Ω_R.

If we want a re-collapse, we must have $\dot{a} = 0$ at some time, in other words:

$$\Omega_\Lambda x^3 + \Omega_K x + \Omega_M = 0.$$

The fate of the universe

Discussion of

$$\Omega_\Lambda x^3 + \Omega_K x + \Omega_M = 0$$

- We know that, for $x = 1$, this expression is +1
- For $\Omega_\Lambda < 0$, for sufficiently large x, the expression will become negative $\Rightarrow$ must have a zero
- For $\Omega_\Lambda = 0$, we know recollapse for $\Omega_M > 1$, requiring $K = +1$
- For $\Omega_\Lambda > 0$, recollapse if Ω_K sufficiently negative (again, $K = +1$).

Overview of FLRW solutions

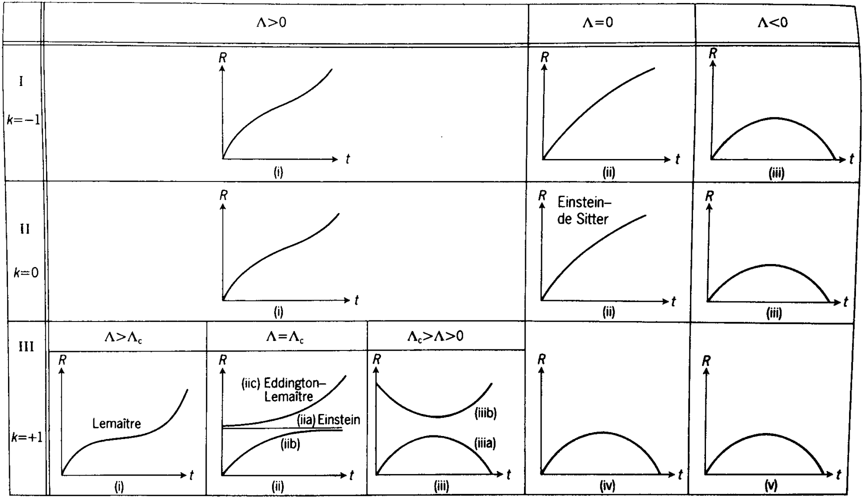
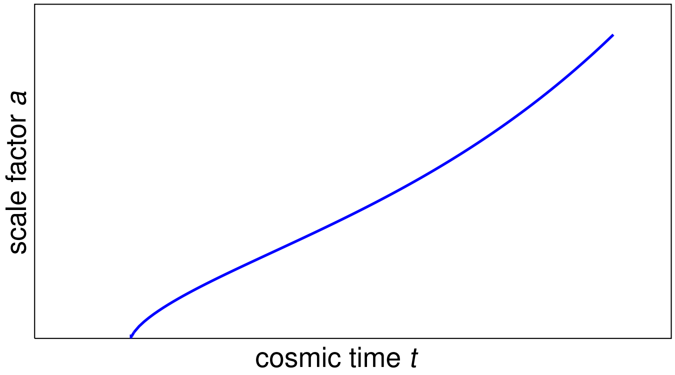


Image from: d'Inverno, *Introducing Einstein's Relativity*, ch. 22.3

FLRW model with $K = 0, \Lambda > 0$

This is the special case that is probably our own universe:

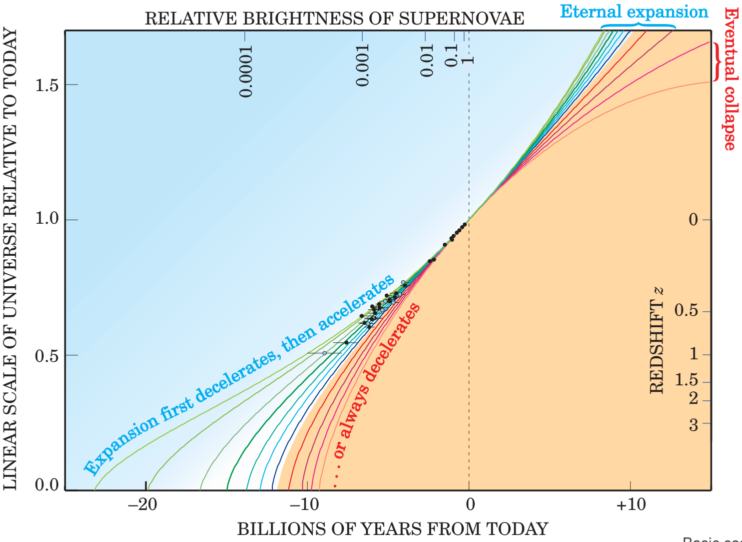


Observing the expanding universe

- Age of the universe (current best: 13.8 billion a) – upper limit!
- Tracing $a(t)$ gives H_0 and constraints on Ω parameters
- Early universe (nucleosynthesis, cosmic background radiation)
- Geometry from CMB and large-scale structure
- Consistency checks: SN time dilation, surface brightness, energy inventory

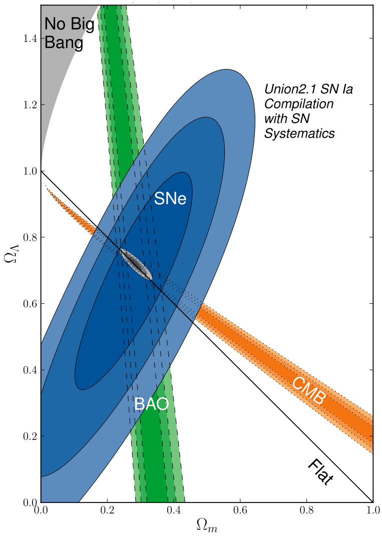
Reconstructing cosmic history

Perlmutter, *Physics Today* 2003



Supernova Cosmology Project Plot

Suzuki et al. 2011



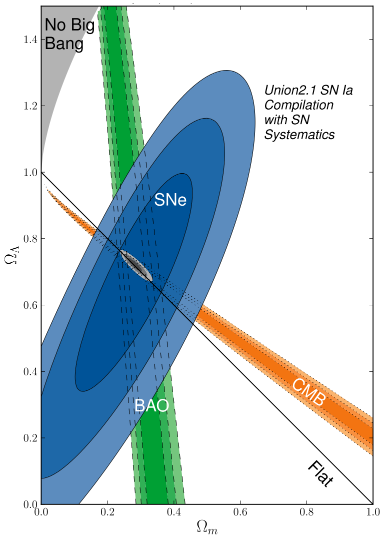
...directly related to our Ω formulae:

Sensitivity of SNe and CMB observations directly linked to

$$q_0 = \frac{1}{2}(\Omega_M - 2\Omega_\Lambda + 2\Omega_R).$$

and

$$\Omega_\Lambda + \Omega_M + \Omega_R + \Omega_K = 1.$$



The matter content of our universe

$$\Omega_M = \left\{ \begin{array}{rcr} \Omega_b & = & 4.9\% \\ \Omega_d & = & 26.8\% \end{array} \right\} = 31.7\%$$

$$\Omega_r = 0.005\%$$

$$\Omega_\Lambda = 68.3\%$$

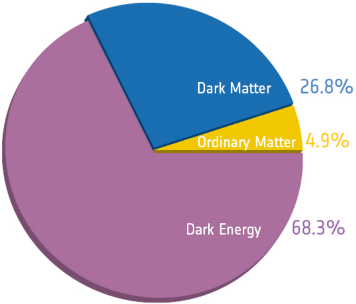


Image: ESA/Planck Collaboration

Where Ω_b = ordinary matter (protons, neutrons, . . .),
Ω_d = dark matter (no interaction with light)
Ω_Λ = (accelerating) dark energy

Literature

d'Inverno, Ray: *Introducing Einstein's Relativity*. Oxford University Press 1992.

Weinberg, Steven: *Cosmology*. Oxford University Press 2008

Wright, Ned: *Cosmology tutorial* at
http://www.astro.ucla.edu/ wright/cosmolog.htm