The hot, early universe

Cosmology Block Course 2014

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Thermodynamics & statistics in an FLRW universe

• Up to now, matter in our universe has not interacted
• If we get back to sufficiently small $a(t)$ (as we must $\rightarrow$ singularity theorems!), we cannot have had separate galaxies
• Early universe: filled with plasma, colliding particles (atoms and photons, nucleons and nucleons) $\Rightarrow$ we need a description from thermodynamics and statistical physics!
Thermodynamics is simple when a system is in thermal equilibrium, and complicated when it isn’t.

(If not in equilibrium: fluid dynamics plus reaction kinetics – can be horribly complicated!)

In equilibrium, certain thermodynamical quantities can be introduced, which take on constant values throughout the system. Best-known of those: Temperature $T$ and pressure $p$.


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Thermodynamics: Macrostates specified by thermodynamic variables like $E, V, T, p, N$.

Statistical mechanics: Microstates of particles (e.g. $N$ particles making up a gas – each has a given momentum at a given time)

Entropy as a quantity to count microstates compatible with a macrostate:

$$S = k \cdot \log \Omega(E, V, N).$$
Systems in equilibrium

Entropy difference in terms of changing variables:

\[ dS = \frac{1}{T} \cdot dE + \frac{p}{T} \cdot dV \]

(this can be taken as *definitions* of *T* and *p*).

Re-write as first law of thermodynamics:

\[ dE = T \cdot dS - p \cdot dV \]
Second law of thermodynamics: $\delta S \geq 0$, but never $\delta S < 0$. Entropy cannot decrease.

Two systems in contact so that $S = S_1 + S_2$, $V = V_1 + V_2$, $E = E_1 + E_2$:

$$dE_1 = -dE_2; \ dV_1 = -dV_2 \text{ so that}$$

$$dS = \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \cdot dE_1 + \left( \frac{p_1}{T_1} - \frac{p_2}{T_2} \right) \cdot dV_1 \geq 0.$$

Second law means: at constant volume, $dE_1 < 0$ if $T_1 > T_2$. At constant temperature, $dV_1 > 0$ if $p_1 > p_2$. In thermodynamics equilibrium, $dS = 0$, so $T_1 = T_2$, $p_1 = p_2$. All as expected.
Entropy density

Define the entropy density $s(T)$ by $S(T, V) = V \cdot s(T)$
(This works because entropy is extensive!).

Then for any adiabatic change,

$$
\begin{align*}
\text{d}(s(T)V) &= V \text{d}s(T) + s(T)\text{d}V = \text{d}S(T, V) = \frac{\text{d}(\rho c^2 V) + p\text{d}V}{T} \\
&= \frac{Vc^2}{T} \frac{\text{d}\rho}{\text{d}T} \text{d}T + \frac{(\rho c^2 + p)\text{d}V}{T}
\end{align*}
$$

This can only hold generally if the coefficients for $\text{d}V$ are equal:

$$s(T) = \frac{\rho c^2 + p}{T}$$

(coefficients for $\text{d}T$ give energy conservation).
Additional contribution to entropy:

\[ dS = \frac{1}{T} \cdot dE + \frac{p}{T} \cdot dV - \sum_i \frac{\mu_i}{T} dN_i \]

(this can be taken as the definition of the \( \mu_i \)). New first law:

\[ dE = T \cdot dS - p \cdot dV + \sum_i \frac{\mu_i}{T} dN_i \]
Chemical potential

Within the same system, in thermal equilibrium, reactions changing particle species 1 into 2 (and other way around), with $N_1 + N_2 = N = \text{const.}$:

$$dS = \frac{1}{T}(\mu_2 - \mu_1)dN_1.$$ 

If $\mu_2 > \mu_1$, number of 1-particles increases! In full thermodynamic equilibrium, from $dS = 0$, $\mu_1 = \mu_2$.

Distinguish between

- **thermal equilibrium** ($T, p$ constant, $\mu_i$ can differ from equilibrium values)
- **chemical equilibrium** ($\mu_i$ have equilibrium values, $T, p$ could differ)
- **thermodynamic equilibrium** ($T, p, \mu_i$ all have equilibrium values)
Multi-particle reactions:

Particle reaction

\[ 1 + 2 \leftrightarrow 3 + 4 \]

(z.B. \( H + \gamma \leftrightarrow p + e^- \), or nuclear reaction):

\[ dN_1 = dN_2 = -dN_3 = -dN_4, \] then in thermal (not necessarily chemical!) equilibrium:

\[ dS = \frac{1}{T}(\mu_3 + \mu_4 - \mu_1 - \mu_2)dN_1 \geq 0. \]

In equilibrium,

\[ \mu_3 + \mu_4 = \mu_1 + \mu_2 \]

(more generally: one such relation for each conserved quantum number: baryon number, lepton number, \ldots)
Particles in thermal equilibrium

Grand-canonical example: E, V, N given — what is the equilibrium state? (Sum over quantum states, treat bosons and fermions differently).

Number density in momentum space cell $d^3p = dp_x \cdot dp_y \cdot dp_z$:

$$n(p_x, p_y, p_z) = \frac{g}{\exp \left( \frac{[E(p) - \mu]}{kT} \right) \mp 1} \frac{d^3p}{\hbar^3}$$

with $E(p) = \sqrt{mc^2 + (pc)^2}$.

Integrate up to get total particle number density!
Highly relativistic particles 1/2

\[ kT > mc^2, \quad kT > \mu, \quad E \approx pc: \]

Equipped with these formulae, it is straightforward to show that

\[ n = \frac{8\pi}{(ch)^3} \zeta(3) g(kT)^3 \cdot \left\{ \begin{array}{c} 1 \text{ bosons} \\ 3/4 \text{ fermions} \end{array} \right. \]

with \( \zeta(3) = 1.2020569031 \). The density for highly relativistic particles is

\[ \rho c^2 = \frac{4\pi^5}{15(ch)^3} g(kT)^4 \cdot \left\{ \begin{array}{c} 1 \text{ bosons} \\ 7/8 \text{ fermions} \end{array} \right. \]

For bosons, this is *Bose-Einstein statistics*, for fermions, *Fermi-Dirac statistics*. 
Highly relativistic particles 2/2

Pressure:

\[ p = \frac{1}{3} \rho c^2 \text{ (as for radiation!)} \]

Entropy density:

\[ s(T) = \frac{4}{3} \frac{\rho c^2}{T} = \frac{16\pi^5}{45(ch)^3} \ gk(kT)^3 \cdot \begin{cases} 
1 & \text{bosons} \\
7/8 & \text{fermions}
\end{cases} \]

Note that the chemical potential \( \nu \) features nowhere in here – for highly relativistic particles, lots of particle-antiparticle pairs flying around, the chemical potential can be neglected!
Thermal photon gas: $g = 2$ (two polarizations), bosons, $m = 0$, $E = h\nu$, in thermal equilibrium $\mu = 0$:

Number density of photons with frequencies between $\nu$ and $\nu + d\nu$ is

$$\frac{8\pi\nu^2/c^2}{\exp(h\nu/kT) - 1}d\nu$$
Thermo & statistic  Back in time  Primordial nucleosynthesis  Cosmic background radiation

Photons

\[ \rho c^2 = \frac{8\pi h}{c^3} \int_{0}^{\infty} \frac{\nu^3}{\exp(h\nu/kT) - 1} \, d\nu \]

From the bosonic energy distribution, it follows that the number \( n(m) \) of photons with energies greater or equal to \( m \cdot kT \) is

\[ n(m) = \frac{n}{2\zeta(3)} \int_{m}^{\infty} \frac{x^2 \, dx}{\exp(x) - 1}. \]

For instance, \( 10^{-9} \) of the photons have energies > \( 26kT \), while \( 10^{-10} \) have energies > \( 29kT \).
Non-relativistic particles

For non-relativistic particles, $mc^2 \gg kT$ and

$$E(p) \approx mc^2 + \frac{p^2}{2m}.$$  

The number density is

$$n = \frac{g}{\hbar^3} (2\pi m kT)^{3/2} \exp\left(-\frac{mc^2 - \mu}{kT}\right)$$

even when not in chemical equilibrium, the energy density is

$$\rho c^2 = n \cdot \left(mc^2 + \frac{3}{2} kT\right)$$

and the pressure

$$p = nkT.$$  

These last expressions are as expected.
How many particle interactions (collisions) in a given situation? For non-relativistic particles in thermal (not necessarily chemical!) equilibrium, number densities $n_1$ and $n_2$, reduced mass $\mu = m_1 m_2 / (m_1 + m_2)$, the collision rate density $C$ is

$$C = n_1 n_2 \langle u \sigma(E) \rangle_u,$$

where the averaging is over a Maxwell-Boltzmann distribution for the relative velocity $u$, and $\sigma(E)$ is the cross section (= collision probability), with $E(u)$ the (velocity-dependent!) energy,

$$\langle u \sigma(E) \rangle_u = 4\pi \left( \frac{\mu}{2\pi kT} \right)^{3/2} \int_0^\infty \exp \left( \frac{-\mu u^2}{2kT} \right) \sigma(E) u^3 \, du.$$
If $\sigma(E)$ is independent, or weakly dependent on $E$, the integral becomes

$$
\langle u\sigma \rangle_u = \sigma \langle u \rangle_u = \sigma \sqrt{\frac{8kT}{\pi \bar{\mu}}}
$$

If one of the particle species is photons, we will approximate the collision rate by

$$
C = n_1 n_2 \sigma c,
$$

scaling, if necessary, with the fraction of photons with an energy larger than the reaction we’re interested in.
The number of reactions per particle of species 1 is

$$\Gamma = \frac{C}{n_1} = n_2 \langle u\sigma(E) \rangle_u,$$

which has physical dimension 1/time.

We compare this with the Hubble parameter

$$H(t) = \frac{\dot{a}}{a}$$

which is the ratio of the change of $a$ to $a$ itself, and thus a measure for the time it takes $a$ to change significantly.
Local equilibrium vs. freeze-out

1. $\Gamma \gg H$: for reactions that establish thermal equilibrium: Local Thermal Equilibrium (LTE): Adiabatic (=isentropic) change from one temperature-dependent equilibrium to the next.

2. $H \gg \Gamma$: Freeze-out – particle concentrations remain constant (or change because of decay, or alternative reactions). Temperature decouples.
For small $x = a/a_0$: Radiation dominates!

Relative scale factor $a/a_0$

Density/present density

Radiation

Dark energy

Dust

The hot, early universe
Radiation dominates

For early times, assume that the only significant contribution comes from radiation:

\[ a = a_0 \sqrt{2 \sqrt{\Omega_{r0} H_0} t} \]

so with \( \Omega_{r0} = 5 \cdot 10^{-5} \) and \( H_0 = 2.18 \cdot 10^{-18}/s \),

\[ a = a_0 \cdot (1.76 \cdot 10^{-10}) \sqrt{\frac{t}{1 \text{ s}}} \]

Hubble parameter goes as

\[ H(t) = \frac{1}{2t} \]
For the phase directly following radiation dominance

For later times assume the only significant contribution comes from the matter density,

\[ a = a_0 \left( \frac{3}{2} \sqrt{\Omega_{m0} H_0 t} \right)^{2/3}, \]

so with \( \Omega_{m0} = 0.317 \) and \( H_0 = 2.18 \cdot 10^{-18}/s \),

\[ a = a_0 \cdot (1.5 \cdot 10^{-12}) \left( \frac{t}{1 \text{ s}} \right)^{2/3}. \]

Hubble parameter goes as

\[ H(t) = \frac{2}{3t}. \]
How does this compare to the exact treatment?

... works except for $t = 8 \cdot 10^{10} - 2 \cdot 10^{14}$ s = 6k – 300k years.
How does the energy distribution evolve?

At some time \( t_1 \), scale factor value \( a(t_1) \), in some volume \( V_1 \), let the photon number between \( \nu_1 \) and \( \nu_1 + d\nu_1 \) be

\[
V_1 \frac{8\pi(\nu_1)^2/c^3}{\exp(h\nu_1/kT_1) - 1} d\nu_1.
\]

At some later time \( t_2 \), the same photons are now spread out over a volume \( V_2 = (x_{21})^3 \) with \( x_{21} = a(t_2)/a(t_1) \). They have been redshifted to \( \nu_2 = \nu_1 / x_{21} \), and their new frequency interval is \( d\nu_2 = d\nu_1 / x_{21} \).
How does the energy distribution evolve?

We can re-write the new number density in terms of the new frequency and interval values $\nu_2$ and $d\nu_2$; the $x_{21}$ mostly cancel, which gives a new number density in the frequency interval $\nu_2 \ldots \nu_2 + d\nu_2$ at time $t_2$ that is

$$\frac{8\pi(\nu_2)^2/c^3}{\exp(h\nu_2 x_{21}/kT_1) - 1}\,d\nu_2.$$ 

This corresponds to the number of photons we would expect in the given frequency range for thermal radiation with temperature

$$T_2 = T_1 \cdot a(t_1)/a(t_2),$$

which for $a(t_2) > a(t_1)$ corresponds to lower temperature.

Temperature scales as $\sim 1/a(t)$! Radiation remains Planckian!
Now it becomes important that the number of photons is so much larger than the baryon number (as you will estimate in the exercise):

$$\eta = \frac{n_B}{n_\gamma} \approx 6 \cdot 10^{-10}.$$

Everything that’s going on will take place in a photon bath! Even absorption reactions hardly matter – they will change the bath by at most $10^{-9}$!
Temperature effects in the early universe

- 2.7 kT (average boson energy)
- 27 kT (a billionth of all photons)

**Photon energy over time**

Cosmic background radiation

Primordial nucleosynthesis

Thermo & statistic

Back in time

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Temperature effects in the early universe

- Hydrogen ionization: 13.6 eV
- Pu ionization: 120 keV

Photon energy over time

Log-log graph showing the photon energy over cosmic time. The graph includes two lines:
- Blue line: 2.7 kT (average boson energy)
- Green line: 27 kT (a billionth of all photons)

The graph shows the decrease in photon energy over time, with cosmic times ranging from $10^{-5}$ to $10^{17}$ seconds.
Photon energy over time

Temperature effects in the early universe

Ni-56 binding energy: 8.8 MeV/nucleon

Electron pair production: 1.2 MeV

2.7 kT (average boson energy)

27 kT (a billionth of all photons)

Cosmic time [s]

Photon energy [eV]
You will calculate some reaction numbers (photon colliding with atom, or with nucleus) in the exercises today. Collision rates will not be a problem — as long as the photons carry sufficient energy to trigger a reaction (ionization, splitting a nucleus...)!
The big picture

Big bang

- $10^{-33}$ s
- Inflation

$10^{-6}$ s
- Quark confinement

1 to 3 min
- Light elements

380,000 a
- CMB

1e8 a
- Galaxies

Now
- 13.8e9 a

The hot, early universe

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In this lecture, we trace temperature back until we have a sea of single nucleons (protons and neutrons). We leave earlier phases (inflation etc.) for later, namely for the last lecture.
As we have seen, at about 1s of cosmic time, sufficient photon energy to tear apart the most stable nuclei (Ni-56). Three-particle reactions much too uncommon (will occur in stars, but not here!), so nuclei have to be built from two-particle reactions.
Key: There is no stable element with $A = 5!$ (Fig. from Coc 2012)
When can nucleosynthesis start?

All nucleosynthesis starts with deuterium production. Binding energy of Deuterium: $2.2\, MeV$.

Nothing happens until the photon energy $27kT$ (this or more carried by $6 \cdot 10^{-10}$ of all photons!) goes below $2.2\, MeV$, which is $(k = 8.6 \cdot 10^{-5}\, eV/K)$ at

$$T_D = 9.5 \cdot 10^8\, K$$

or

$$\frac{a}{a_0} = \frac{T_0}{T_D} = 3 \cdot 10^{-9}$$

so

$$t = 290\, s.$$  

... at which time, all neutrons are quickly built into $^4He$; most stable configuration, gap at $A = 5$! But how many neutrons do we have in the first place?
How many neutrons do we have to start with?

Weak interactions between protons and neutrons:

\[ n + \nu_e \leftrightarrow p + e^- \]

Reaction rate for these weak interactions is:

\[ \sigma_w = 10^{-47} m^2 \left( \frac{kT}{1 \text{ MeV}} \right)^2 \]

Estimating the reaction rate (similar to exercise), this works only while there is still pair production, and stops at

\[ t \approx 1s, \quad kT \approx 0.8 \text{ MeV}. \]
How many neutrons do we have to start with?

\[ n + \nu_e \leftrightarrow p + e^- \]

in equilibrium above or about \( kT = 1 \text{ MeV} \) means that

\[ \mu_n + \mu_{\nu_e} = \mu_p + \mu_{e^-} \quad \Rightarrow \quad \mu_n = \mu_p \]

(where, as mentioned above, we have neglected \( \mu \) for highly relativistic, pair-produced particles. But the non-relativistic number density was)

\[
n = \frac{g}{\hbar^3} (2\pi m kT)^{3/2} \exp \left( -\frac{mc^2 - \mu}{kT} \right),
\]

so

\[
\frac{n_n}{n_p} = \left( \frac{m_n}{m_p} \right)^{3/2} \exp \left( -\frac{(m_n - m_p)c^2}{kT} \right).
\]
How many neutrons do we have to start with?

\[
\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left(-\frac{(m_n - m_p)c^2}{kT}\right).
\]

Inserting \(kT = 0.8 \text{ MeV}\) and \((m_n - m_p)c^2 = 1.293 \text{ MeV}\), neglecting the pre-factor:

\[
\frac{n_n}{n_p} = 0.198 \approx \frac{1}{5}
\]

at \(t = 1\text{s}\).

Problem: Neutrons decay, with half-life 611 s! Between \(t = 1\text{ s}\) and \(t = 290\text{ s}\), the number ratio has dropped from 1 neutron to 5 protons to

\[
(1/2)^{290\text{ s}/611\text{ s}} \approx 0.72 \text{ neutrons per proton} \approx \frac{1}{7}.
\]
Helium fraction

With 2 neutron per 14 protons (1/7), you can make $1 \ ^4\text{He}$ plus 12 protons. Mass ratio between He and total mass is

$$Y = \frac{4}{16} = 25\%$$

...this is the fairly robust main prediction for big bang nucleosynthesis!
3.2. Improvement of Some Critical Reaction Rates

### 3.2.1. $^7\text{Li}(t, n)_{^9\text{Be}}$ Affecting $^9\text{Be}$ and $^9\text{Be}$

This factor corresponds to the ratio between the $^7\text{Li}(t, n)_{^9\text{Be}}$ (which includes all neutron final states) and $^7\text{Li}(t, n_0)_{^9\text{Be}}$ cross section (where $n_0$ denotes transitions to the ground state of $^9\text{Be}$ only) by Brune et al. (1991). The upper limit is assumed to be a factor of 25 larger than the lower limit.

The final rate with the estimated uncertainties are shown and compared with the TALYS and Boyd et al. (1993) rates in Figure 18.

#### 3.2.2. $^7\text{Li}(d, n)_{^4\text{He}}$ Affecting CNO

In this section, the above-mentioned critical reactions are analyzed and their rates re-evaluated on the basis of suited reaction models. In addition, realistic uncertainties affecting these rates are estimated in order to provide realistic predictions.

The total reaction rate consists of two contributions, namely a resonance and a direct part. The direct contribution is obtained by a numerical integration with a constant $S$ factor (Schmid et al. 1993). The corresponding upper and lower limits are estimated by multiplying the TALYS predictions in Figure 16 with the estimated uncertainties.

More precisely, the lower limit of the total reaction rate is obtained from the theoretical analysis of Yamamoto et al. (1993) based on the experimental determination of the $^7\text{Li}(d, n)_{^4\text{He}}$ rate, experimental data from Brune et al. (1991) as well as theoretical calculations from Yamamoto et al. (1993) are considered.

The recommended rate is obtained by the geometrical means of the lower and upper limits of the total reaction rate. The resonance part is calculated by Equations (11) and (14) in the NACRE evaluation (Angulo et al. 1999) where the resonance parameters are estimated on the basis of Equations (11) and (14) in the NACRE evaluate $\Omega_B h^2 = \text{WMAP}$.

#### 3.2.3. $^7\text{Li}(d, \gamma)_{^7\text{Be}}$ Affecting $^7\text{Be}$

The lower limit is estimated by multiplying the TALYS predictions in Figure 16 by a $-factor$ by a $-factor$ of $10$ and $0.1$, respectively. The resonance contribution is estimated using the $S$-factor (Schmid et al. 1993). The corresponding upper and lower limits are estimated by multiplying the TALYS predictions in Figure 16.
Comparing with observations

Figure 3. Abundances of $^4$He (mass fraction), D, $^3$He and $^7$Li (by number relative to H) as a function of the baryon over photon ratio $\eta$ (or $\Omega_B h^2$) showing the effect of nuclear uncertainties [31]. The vertical stripe corresponds to the WMAP baryonic density [5] while the horizontal area represent the adopted primordial abundances (dotted lines those adopted in CV10 [31]). The dashed curves represent previous calculations [29] before the re-evaluation [25] of the $^3$He($\alpha$, $\gamma$)$^7$Be rate. The dot-dashed lines corresponds to 4 effective neutrino families.
How do we get from the plasma state (hydrogen and helium nuclei, electrons, photons) to an atomic, transparent universe?

Reaction:

\[ H + \gamma \leftrightarrow p + e^- \]

As you will estimate in the exercises (at least the LHS), lots and lots of collisions – system will be in equilibrium!

In thermodynamic equilibrium, thermodynamic potentials add up:

\[ \mu_H + \mu_\gamma = \mu_p + \mu_e^- . \]

... but \( \mu_\gamma \) in chemical equilibrium is zero!
But there are atomic reactions where either one or two photons can be produced:

\[ A \rightarrow B + \gamma \quad \text{versus} \quad A \rightarrow B^* + \gamma, \quad B^* \rightarrow A + \gamma. \]

This means

\[ \mu_A = \mu_B + \mu_\gamma, \]

but also

\[ \mu_A = \mu_{B^*} + \mu_\gamma = \mu_B + 2\mu_\gamma \]

\[ \Rightarrow \text{this can only hold if } \mu_\gamma = 0! \]
Equilibrium state for ionization

Since $\mu_\gamma = 0$, in equilibrium for the reaction $H + \gamma \leftrightarrow p + e^-$,

$$\mu_H = \mu_p + \mu_{e^-}.$$

Our particles are all non-relativistic, so

$$n = \frac{g}{h^3}(2\pi m kT)^{3/2} \exp\left(-\frac{mc^2 - \mu}{kT}\right),$$

and with the $g_e = g_p = 2$ (spin $\pm 1/2$) and $g_H = 1 + 3 = 4$ (spin 0 plus spin 1), so

$$\frac{n_p n_e}{n_H} = \frac{(2\pi m_e kT)^{3/2}}{h^3} \left(\frac{m_p}{m_H}\right)^{3/2} \exp\left(-\frac{B}{kT}\right)$$

where $B = (m_p + m_e - m_H)c^2 = 13.6 \text{ eV}$ is the binding energy.
Equilibrium state for ionization

Charge neutrality means \( n_e = n_p \). Define the ionization fraction

\[
x_e \equiv \frac{n_e}{n_e + n_H},
\]

so that

\[
\frac{x_e^2}{1 - x_e} = \frac{n_e}{n_H(n_e + n_H)} = \frac{n_e}{n_H n_b} = \frac{(2\pi m_e kT)^{3/2}}{n_B h^3} \left( \frac{m_p}{m_H} \right)^{3/2} \exp \left( -\frac{B}{kT} \right)
\]

with \( n_b \) the baryon number density. But the number density is related to the modern value, and the present (CMB) temperature \( T_0 \), as

\[
n_b(t) = n_{b0} \left( \frac{a_0}{a(t)} \right)^3 = n_{b0} \left( \frac{T}{T_0} \right)^3.
\]
Equilibrium state for ionization

Inserting this into the equation, neglecting the $m_p/m_H$ term,

\[
\frac{x_e^2}{1 - x_e} = \frac{1}{n_b_0 h^3} \left( \frac{2\pi m_e k}{T} \right)^{3/2} \exp \left( -\frac{B}{kT} \right)
\]

\[
= 8.78 \cdot 10^{21} \left( \frac{1 K}{T} \right)^{3/2} \exp \left( -\frac{1.6 \cdot 10^5 K}{T} \right)
\]
• for recombination to be considered finished, $x \ll 1$ and $x^2/(1-x) \approx x^2$; since $1/\eta$ is a huge number, $kT \ll \chi$ is required for $x$ to be small; for example, putting $x = 0.1$ yields $kT_{\text{rec}} = 0.3 \text{ eV}$, or $T_{\text{rec}} \approx 3500 \text{ K}$ (1.14)

\begin{itemize}
  \item since $\chi = 13.6 \text{ eV}$, one would naively expect $T_{\text{rec}} \approx 10^5 \text{ K}$; the very large photon-to-baryon ratio $1/\eta$ delays recombination considerably
  \item strictly, Saha's equation is invalid for cosmological recombination because it assumes thermal equilibrium between the reaction partners, which breaks down as recombination proceeds; however, due to the rapid progress of recombination, the deviation between the ionisation degree predicted by Saha's equation and by an exact treatment remains small
\end{itemize}

Two-Photon Recombination

- direct hydrogen recombination produces energetic photons; the final transition to the ground state is Lyman-$\alpha$ ($2^P \rightarrow 1^S$), so that the energy of the emitted photon is $h\nu \geq E_{\text{Ly} \alpha} = \frac{3\chi}{4} = 10.2 \text{ eV}$
- the abundant Ly-$\alpha$ photons keep reionising the cosmic gas because they cannot stream away as from hydrogen clouds; the energy loss due to cosmic expansion is slow
- recombination can only proceed by production of photons with lower energy than Ly-$\alpha$; this is possible through the forbidden transition $2^S \rightarrow 1^S$, which requires the emission of two photons
- this process is slow, hence recombination proceeds at a somewhat lower rate than predicted by Saha's equation

Thickness of the Recombination Shell

- recombination is not instantaneous, but requires a finite time interval; there is thus a "recombination shell" with finite thickness
- the optical depth along a light ray through the recombination shell is
\[ \tau = \int n_e \sigma_T d r = \frac{n_B \sigma_T}{\dot{a}} \int x \, d r \quad (1.145) \]

where $\sigma_T$ is the Thomson scattering cross section,
\[ \sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{m_e c^2} \right)^2 = 6.65 \times 10^{-25} \text{ cm}^2 \quad (1.146) \]

and $dr = c dt = c da/\dot{a}$ is the proper length interval.
Recombination at what redshift?

From the previous graph, \( T_{\text{rec}} \approx 0.3 \text{ eV} \approx 3500 \text{ K} \).

By scaling behaviour of \( T \):

\[
\frac{a(t_{\text{rec}})}{a_0} = \frac{T_0}{T_{\text{rec}}} = 7.8 \cdot 10^{-4} = \frac{1}{1 + z}
\]

so

\[ z \approx 1280. \]

Using the “matter only” approximation

\[
a = a_0 \left( \frac{3}{2} \sqrt{\Omega_m H_0 t} \right)^{2/3}
\]

we get

\[ t_{\text{rec}} = 376,000 \, a. \]
Precision CMB: COBE-FIRAS (Mather et al.)

Data from Fixsen et al. 1996
Best Planck fit: $T = 2.728\,\text{K}$
Range shown: spectrum $\pm 3\,\sigma$

Data from Fixsen et al. 1996 via http://lambda.gsfc.nasa.gov
Precision CMB: COBE-FIRAS (Mather et al.)

Data from Fixsen et al. 1996 via http://lambda.gsfc.nasa.gov

Best Planck fit: $T = 2.728$ K

Range shown: spectrum $\pm 100 \sigma$
Precision CMB: COBE-FIRAS (Mather et al.)

Data from Fixsen et al. 1996
Best Planck fit: $T = 2.728 \text{ K}$
Range shown: spectrum $\pm 500 \sigma$

Data from Fixsen et al. 1996 via http://lambda.gsfc.nasa.gov

The hot, early universe
The big picture

Back in time

Primordial nucleosynthesis
Cosmic background radiation

Thermo & statistic

Big bang

10^{-33} s
Inflation

1e^{-6} s
Quark confinement

1 s to 3 min light elements

Cosmic background radiation

Radiation era

Matter era

1e8 a galaxies

380,000 a CMB

Now

13.8e9 a

The hot, early universe

Simon Glover & Markus Pössel
Little, Andrew J.: *An Introduction to Modern Cosmology*. Wiley 2003 [brief and basic]


Weinberg, Steven: *Cosmology*. Oxford University Press 2008 [advanced]

Weinberg, Steven: *Gravitation and Cosmology*. Wiley & Sons 1972 [advanced and detailed]